Calculus & Analytic Geometry II

Separation of Variables

Big Idea. For nice differential equations, we can sometimes find a simple, closed form solution.

Recall Example. Suppose $y' = y + 1$ and $y(0) = 1$. We estimated $y(1)$ using $\Delta t = \frac{1}{3}$ and found $y(1) \approx 3.741$. Creating a spreadsheet makes this computation more palatable.

Summary of data

dt	$y(t)$
1/3	3.741
.1	4.187
.01	4.410
.001	
.0001	

Verify that $y(t) = 2e^t - 1$ is a solution to the given initial value problem.

Consider the differentials that appear in Leibnitz notation as variables that can be manipulated. Can you isolate all the ys on one side and the ts on the other?

$$\frac{dy}{dt} = y + 1$$

You have just completed your first analytical solution using the method of *separation of variables.*

More Practice.

$$\begin{cases} y' &= -2y \\ y(0) &= 1 \end{cases} \qquad \begin{cases} \frac{dH}{dt} &= -k(H-20) \\ H(0) &= 30 \end{cases} \qquad \begin{cases} \frac{dP}{dt} &= 2P - 2Pt \\ P(0) &= 5 \end{cases}$$

$y = e^{-2t}$
$H = 20 + 10e^{-kt}$
$P = 5e^{2t-t^2}$

Question. What property makes these *nice* differential equations?

Justification for Separation of Variables

Suppose $\frac{dy}{dx} = g(x)f(y)$ and $f(y) \neq 0$. Let $h(y) = 1/f(y)$ and rewrite

Integrate both sides with respect to x:

$$\int h(y)\frac{dy}{dx}dx = \int g(x)dx$$

Substitute using
$u = y(x)$
$du =$

$$\int h(y)dy = \int g(x)dx$$

Another case where the Leibniz notation makes the chain rule look like cancellation.

Example. On page 42, we showed that $P(t) = \frac{1}{1+2e^{-kt}}$ was a one solution to the differential equation $\frac{dP}{dt} = kP(1-P)$. Can we now see why?

Problem. A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour? ($\approx$ 4.9%)

Revisit Example 9.1.1, pg. 570. Find the general solution to $y' = \frac{1}{2}(y^2 - 1)$.

$(1 \pm ce^x)/(1 \mp ce^x)$

Calculus & Analytic Geometry II

Models of Population Growth

Natural Growth. $\frac{1}{P}\frac{dp}{dt} = k$

Logistic Growth. $\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{K}\right)$

What happens to the population as $t \to \infty$? What happens to the population if $P > K$, P is small compared to K, $P < K$ but close to K?

Application. A lake is stocked with 400 fish that is estimated to have a carrying capacity of 10,000 fish. If the number of fish tripled in the first year

- model population of fish at time t.
- How long will it take for the population to increase to 5000?

Doomsday. A differential equation of the form $\frac{dy}{dt} = ky^{1+c}$ where c and k are positive constants, is called a *doomsday equation*. Let's see if we can find out why.

- Find a general solution satisfying the initial condition $y(0) = y_0$.
- Show that there is a finite time T (doomsday!) where $\lim_{t\to T^-} y(t) = \infty$.
- A concrete example. Two rabbits beget 14 rabbits after 3 months. If $y' = ky^{1.01}$, when is doomsday?

Modified Logistic version 1. $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - 15$

- Suppose that $P(t)$ represents fish population at time t. What is the meaning of -15?
- Draw a direction field for this differential equation. Sketch several solution curves.
- What are the equilibrium solutions?
- Can you solve this system explicitly? (Use initial populations of 200 and 300).

Modified Logistic version 2. $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$

- Draw a direction field for this differential equation. Sketch several solution curves.
- What are the equilibrium solutions?
- Can you solve this system explicitly?
- What happens to the population if $P(0) < 200$?