4. Let $u = x$, $dv = e^{-x}\,dx \Rightarrow du = dx$, $v = -e^{-x}$. Then $\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$.

7. Let $u = x^2$, $dv = \sin\pi x\,dx \Rightarrow du = 2x\,dx$ and $v = -\frac{1}{\pi}\cos\pi x$. Then

$I = \int x^2 \sin\pi x\,dx = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\int x\cos\pi x\,dx$ $(\star)$. Next let $U = x$, $dV = \cos\pi x\,dx \Rightarrow dU = dx$,

$V = \frac{1}{\pi}\sin\pi x$, so $\int x\cos\pi x\,dx = \frac{1}{\pi}x\sin\pi x - \frac{1}{\pi}\int\sin\pi x\,dx = \frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1$.

Substituting for $\int x\cos\pi x\,dx$ in $(\star)$, we get

$I = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\left(\frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1\right) = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x + \frac{2}{\pi^3}\cos\pi x + C$, where $C = \frac{2}{\pi}C_1$.

10. Let $u = \sin^{-1}x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then $\int\sin^{-1}x\,dx = x\sin^{-1}x - \displaystyle\int\frac{x}{\sqrt{1-x^2}}\,dx$. Setting

$t = 1 - x^2$, we get $dt = -2x\,dx$, so $-\displaystyle\int\frac{x\,dx}{\sqrt{1-x^2}} = -\int t^{-1/2}\left(-\tfrac{1}{2}\,dt\right) = \tfrac{1}{2}(2t^{1/2}) + C = t^{1/2} + C = \sqrt{1-x^2} + C$.

Hence, $\int\sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$.

13. Let $u = t$, $dv = \sec^2 2t\,dt \Rightarrow du = dt$, $v = \frac{1}{2}\tan 2t$. Then

$\int t\sec^2 2t\,dt = \frac{1}{2}t\tan 2t - \frac{1}{2}\int\tan 2t\,dt = \frac{1}{2}t\tan 2t - \frac{1}{4}\ln|\sec 2t| + C$.

17. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \Rightarrow du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then

$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \Rightarrow dU = -3\sin 3\theta\,d\theta$,

$V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \Rightarrow$

$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

23. Let $u = \ln x$, $dv = x^{-2}\,dx \Rightarrow du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2\frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

47. Let $u = (\ln x)^n$, $dv = dx \Rightarrow du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$\int(\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$.

50. Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx \quad \Rightarrow \quad du = (n-2)\sec^{n-3} x \sec x \tan x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned}\int \sec^n x\,dx &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx\end{aligned}$$

so $(n-1)\int \sec^n x\,dx = \tan x \sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx$. If $n - 1 \neq 0$, then

$$\int \sec^n x\,dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

51. By repeated applications of the reduction formula in Exercise 47,

$$\begin{aligned}\int (\ln x)^3\,dx &= x\,(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int (\ln x)^1\,dx\right] \\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6\left[x(\ln x)^1 - 1\int (\ln x)^0\,dx\right] \\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int 1\,dx = x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C\end{aligned}$$

58. Volume $= \int_0^1 2\pi x(e^x - e^{-x})\,dx = 2\pi \int_0^1 (xe^x - xe^{-x})\,dx = 2\pi\left[\int_0^1 xe^x\,dx - \int_0^1 xe^{-x}\,dx\right]$ [both integrals by parts]

$= 2\pi\left[(xe^x - e^x) - \left(-xe^{-x} - e^{-x}\right)\right]_0^1 = 2\pi[2/e - 0] = 4\pi/e$