

27. $x^2 + 2x = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1$. Let $x + 1 = 1 \sec \theta$,

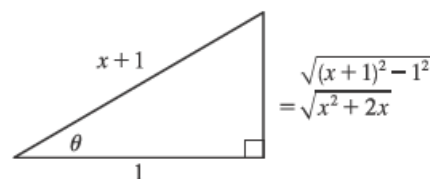
so $dx = \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 + 2x} = \tan \theta$. Then

$$\int \sqrt{x^2 + 2x} dx = \int \tan \theta (\sec \theta \tan \theta d\theta) = \int \tan^2 \theta \sec \theta d\theta$$

$$= \int (\sec^2 \theta - 1) \sec \theta d\theta = \int \sec^3 \theta d\theta - \int \sec \theta d\theta$$

$$= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} (x + 1) \sqrt{x^2 + 2x} - \frac{1}{2} \ln |x + 1 + \sqrt{x^2 + 2x}| + C$$



32. (a) Let $x = a \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$I = \int \frac{x^2}{(x^2 + a^2)^{3/2}} dx = \int \frac{a^2 \tan^2 \theta}{a^3 \sec^3 \theta} a \sec^2 \theta d\theta = \int \frac{\tan^2 \theta}{\sec \theta} d\theta = \int \frac{\sec^2 \theta - 1}{\sec \theta} d\theta$$

$$= \int (\sec \theta - \cos \theta) d\theta = \ln |\sec \theta + \tan \theta| - \sin \theta + C$$

$$= \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| - \frac{x}{\sqrt{x^2 + a^2}} + C = \ln(x + \sqrt{x^2 + a^2}) - \frac{x}{\sqrt{x^2 + a^2}} + C_1$$

(b) Let $x = a \sinh t$. Then

$$I = \int \frac{a^2 \sinh^2 t}{a^3 \cosh^3 t} a \cosh t dt = \int \tanh^2 t dt = \int (1 - \operatorname{sech}^2 t) dt = t - \tanh t + C$$

$$= \sinh^{-1} \frac{x}{a} - \frac{x}{\sqrt{a^2 + x^2}} + C$$

33. The average value of $f(x) = \sqrt{x^2 - 1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1} \int_1^7 \frac{\sqrt{x^2 - 1}}{x} dx = \frac{1}{6} \int_0^\alpha \frac{\tan \theta}{\sec \theta} \cdot \sec \theta \tan \theta d\theta \quad \left[\begin{array}{l} \text{where } x = \sec \theta, dx = \sec \theta \tan \theta d\theta, \\ \sqrt{x^2 - 1} = \tan \theta, \text{ and } \alpha = \sec^{-1} 7 \end{array} \right]$$

$$= \frac{1}{6} \int_0^\alpha \tan^2 \theta d\theta = \frac{1}{6} \int_0^\alpha (\sec^2 \theta - 1) d\theta = \frac{1}{6} [\tan \theta - \theta]_0^\alpha$$

$$= \frac{1}{6} (\tan \alpha - \alpha) = \frac{1}{6} (\sqrt{48} - \sec^{-1} 7)$$

42. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is

$$A = 2 \int_{-5}^2 \sqrt{25 - y^2} dy$$

$$= \left[25 \arcsin(y/5) + y \sqrt{25 - y^2} \right]_{-5}^2 \quad [\text{substitute } y = 5 \sin \theta]$$

$$= 25 \arcsin \frac{2}{5} + 2\sqrt{21} + \frac{25}{2} \pi \approx 58.72 \text{ ft}^2$$

so the fraction of the total capacity in use is $\frac{A}{\pi(5)^2} \approx \frac{58.72}{25\pi} \approx 0.748$ or 74.8%.

