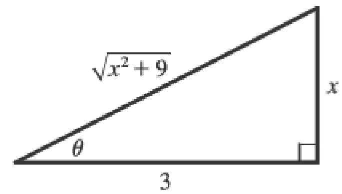
3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9(\tan^2\theta + 1)} = \sqrt{9\sec^2\theta}$$

$$= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.$$



$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta = 3^3\int \tan^3\theta\,\sec\theta d\theta = 3^3\int \tan^2\theta\,\tan\theta\,\sec\theta\,d\theta$$

$$= 3^3\int(\sec^2\theta - 1)\,\tan\theta\,\sec\theta\,d\theta = 3^3\int(u^2-1)\,du \qquad [u = \sec\theta,\ du = \sec\theta\,\tan\theta\,d\theta]$$

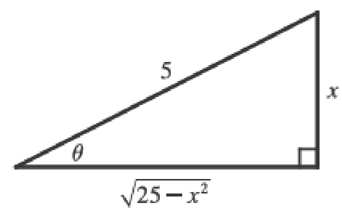
$$= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{(x^2+9)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C$$

$$= \tfrac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C \quad \text{or} \quad \tfrac{1}{3}(x^2-18)\sqrt{x^2+9} + C$$

7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then

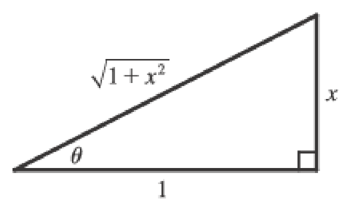
$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}\,5\cos\theta\,d\theta = \frac{1}{25}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{25}\cot\theta + C = -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$



19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$

and $\sqrt{1+x^2} = \sec\theta$, so



$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\,\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}\,(1+\tan^2\theta)\,d\theta$$

$$= \int(\csc\theta + \sec\theta\,\tan\theta)\,d\theta$$

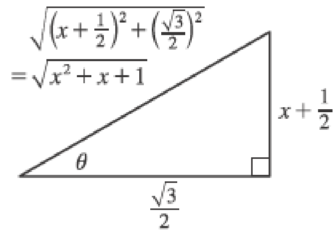
$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.39]}$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

25. $x^2+x+1=\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$. Let

$x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$, so $dx=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and $\sqrt{x^2+x+1}=\frac{\sqrt{3}}{2}\sec\theta$.

Then



$$\begin{aligned}
\int \frac{x}{\sqrt{x^2+x+1}}\,dx &= \int \frac{\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}}{\frac{\sqrt{3}}{2}\sec\theta}\,\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta \\
&= \int\left(\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}\right)\sec\theta\,d\theta=\int\frac{\sqrt{3}}{2}\tan\theta\,\sec\theta\,d\theta-\int\frac{1}{2}\sec\theta\,d\theta \\
&= \frac{\sqrt{3}}{2}\sec\theta-\frac{1}{2}\ln|\sec\theta+\tan\theta|+C_1 \\
&= \sqrt{x^2+x+1}-\frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}+\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right|+C_1 \\
&= \sqrt{x^2+x+1}-\frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\left[\sqrt{x^2+x+1}+\left(x+\frac{1}{2}\right)\right]\right|+C_1 \\
&= \sqrt{x^2+x+1}-\frac{1}{2}\ln\frac{2}{\sqrt{3}}-\frac{1}{2}\ln\left(\sqrt{x^2+x+1}+x+\frac{1}{2}\right)+C_1 \\
&= \sqrt{x^2+x+1}-\frac{1}{2}\ln\left(\sqrt{x^2+x+1}+x+\frac{1}{2}\right)+C,\quad \text{where } C=C_1-\frac{1}{2}\ln\frac{2}{\sqrt{3}}
\end{aligned}$$