7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}dx = \int \left(1+\frac{6}{x-6}\right)dx = x + 6\ln|x-6| + C$

12. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1 = A(x+2) + B(x+1)$. Substituting -2 for x gives $-3 = -B \quad\Leftrightarrow\quad B = 3$. Substituting -1 for x gives $-2 = A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2}\right)dx = \left[-2\ln|x+1| + 3\ln|x+2|\right]_0^1$$
$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \quad\Rightarrow\quad 4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$. Setting $y=0$ gives $-12 = -6A$, so $A = 2$. Setting $y=-2$ gives $18 = 10B$, so $B = \frac{9}{5}$. Setting $y=3$ gives $3 = 15C$, so $C = \frac{1}{5}$.

Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right)dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2$$
$$= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2$$
$$= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

24. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2-x+6 = A(x^2+3) + (Bx+C)x$.

Substituting 0 for x gives $6 = 3A \quad\Leftrightarrow\quad A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \quad\Rightarrow$ $B = 1-2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int\left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx = \int\left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right)dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

30. $\dfrac{3x^2+x+4}{x^4+3x^2+2} = \dfrac{3x^2+x+4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$3x^2+x+4 = (Ax+B)(x^2+2)+(Cx+D)(x^2+1) \quad \Leftrightarrow$

$3x^2+x+4 = (Ax^3+Bx^2+2Ax+2B)+(Cx^3+Dx^2+Cx+D) \quad \Leftrightarrow$

$3x^2+x+4 = (A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$. Comparing coefficients gives us the following system of equations:

$$\begin{aligned} A+C&=0 \quad \textbf{(1)} \qquad & B+D&=3 \quad \textbf{(2)} \\ 2A+C&=1 \quad \textbf{(3)} \qquad & 2B+D&=4 \quad \textbf{(4)} \end{aligned}$$

Subtracting equation (1) from equation (3) gives us $A=1$, so $C=-1$. Subtracting equation (2) from equation (4) gives us $B=1$, so $D=2$. Thus,

$$\begin{aligned} I = \int \frac{3x^2+x+4}{x^4+3x^2+2}\,dx &= \int \frac{x+1}{x^2+1}\,dx + \int \frac{-x+2}{x^2+2}\,dx \\ &= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+2}\,dx + 2\int \frac{1}{x^2+\left(\sqrt{2}\right)^2}\,dx \\ &= \tfrac{1}{2}\ln\left|x^2+1\right| + \tan^{-1}x - \tfrac{1}{2}\ln\left|x^2+2\right| + 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C \\ &= \tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}\ln(x^2+2) + \tan^{-1}x + \sqrt{2}\,\tan^{-1}\left(x/\sqrt{2}\right)+C \end{aligned}$$

36. Let $u = x^5+5x^3+5x$, so that $du = (5x^4+15x^2+5)dx = 5(x^4+3x^2+1)dx$. Then

$$\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx = \int \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\ln|u|+C = \frac{1}{5}\ln\left|x^5+5x^3+5x\right|+C$$

46. Let $u=\sqrt{1+\sqrt{x}}$, so that $u^2 = 1+\sqrt{x}$, $x=(u^2-1)^2$, and $dx = 2(u^2-1)\cdot 2u\,du = 4u(u^2-1)\,du$. Then

$\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du = \int \frac{4u^2}{u^2-1}\,du = \int\left(4+\frac{4}{u^2-1}\right)du$. Now

$\dfrac{4}{u^2-1} = \dfrac{A}{u+1}+\dfrac{B}{u-1} \quad \Rightarrow \quad 4 = A(u-1)+B(u+1)$. Setting $u=1$ gives $4=2B$, so $B=2$. Setting $u=-1$ gives $4=-2A$, so $A=-2$. Thus,

$$\begin{aligned} \int\left(4+\frac{4}{u^2-1}\right)du &= \int\left(4-\frac{2}{u+1}+\frac{2}{u-1}\right)du = 4u-2\ln|u+1|+2\ln|u-1|+C \\ &= 4\sqrt{1+\sqrt{x}} - 2\ln\left(\sqrt{1+\sqrt{x}}+1\right)+2\ln\left(\sqrt{1+\sqrt{x}}-1\right)+C \end{aligned}$$

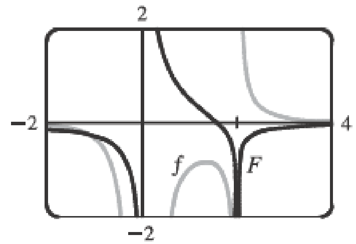
54. $\dfrac{1}{x^3-2x^2} = \dfrac{1}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} \quad \Rightarrow \quad 1 = (A+C)x^2 + (B-2A)x - 2B$, so $A + C = B - 2A = 0$ and

$-2B = 1 \quad \Rightarrow \quad B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $\dfrac{1}{x^3-2x^2}$ is

$$\int \frac{dx}{x^3-2x^2} = -\frac{1}{4}\int\frac{dx}{x} - \frac{1}{2}\int\frac{dx}{x^2} + \frac{1}{4}\int\frac{dx}{x-2}$$

$$= -\tfrac{1}{4}\ln|x| - \tfrac{1}{2}(-1/x) + \tfrac{1}{4}\ln|x-2| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + \frac{1}{2x} + C$$



We plot this function with $C = 0$ on the same screen as $y = \dfrac{1}{x^3-2x^2}$.

60. Let $t = \tan(x/2)$. Then, by Exercise 57,

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt/(1+t^2)}{1 + 2t/(1+t^2) - (1-t^2)/(1+t^2)} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt}{1+t^2+2t-1+t^2}$$

$$= \int_{1/\sqrt{3}}^{1}\left[\frac{1}{t} - \frac{1}{t+1}\right]dt = \Big[\ln t - \ln(t+1)\Big]_{1/\sqrt{3}}^{1} = \ln\frac{1}{2} - \ln\frac{1}{\sqrt{3}+1} = \ln\frac{\sqrt{3}+1}{2}$$

67. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain

$$f(x) = \frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x + 48{,}935)/260{,}015}{x^2+x+5}$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) $$\int f(x)\,dx = \tfrac{24{,}110}{4879} \cdot \tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7| + \frac{1}{260{,}015}\int \frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}\,dx + C$$

$$= \tfrac{24{,}110}{4879} \cdot \tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7| + \tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln\left(x^2+x+5\right) + 37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right] + C$$

$$= \tfrac{4822}{4879}\ln|5x+2| - \tfrac{334}{323}\ln|2x+1| - \tfrac{3146}{80{,}155}\ln|3x-7| + \tfrac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right) + \tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}(2x+1)\right] + C$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879} - \frac{334\ln(2x+1)}{323} - \frac{3146\ln(3x-7)}{80{,}155} + \frac{11{,}049\ln(x^2+x+5)}{260{,}015} + \frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

69. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$\begin{aligned} F(a) &= \lim_{x\to a} F(x) && \text{[by continuity of } F\text{]} \\ &= \lim_{x\to a} G(x) && \text{[whenever } Q(x)\neq 0\text{]} \\ &= G(a) && \text{[by continuity of } G\text{]} \end{aligned}$$