4. $y' = y^2 \sin x \;\Rightarrow\; \dfrac{dy}{dx} = y^2 \sin x \;\Rightarrow\; \dfrac{dy}{y^2} = \sin x\,dx \;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int \sin x\,dx \;\Rightarrow$

$-\dfrac{1}{y} = -\cos x + C \;\Rightarrow\; \dfrac{1}{y} = \cos x - C \;\Rightarrow\; y = \dfrac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.

5. $(1 + \tan y)\,y' = x^2 + 1 \;\Rightarrow\; (1 + \tan y)\dfrac{dy}{dx} = x^2 + 1 \;\Rightarrow\; \left(1 + \dfrac{\sin y}{\cos y}\right) dy = (x^2 + 1)\,dx \;\Rightarrow$

$\displaystyle\int \left(1 - \frac{-\sin y}{\cos y}\right) dy = \int (x^2 + 1)\,dx \;\Rightarrow\; y - \ln|\cos y| = \tfrac{1}{3}x^3 + x + C.$

Note: The left side is equivalent to $y + \ln|\sec y|$.

7. $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1+y^2}} \;\Rightarrow\; y\sqrt{1+y^2}\,dy = te^t\,dt \;\Rightarrow\; \int y\sqrt{1+y^2}\,dy = \int te^t\,dt \;\Rightarrow\; \tfrac{1}{3}\left(1+y^2\right)^{3/2} = te^t - e^t + C$

[where the first integral is evaluated by substitution and the second by parts] $\Rightarrow\; 1 + y^2 = [3(te^t - e^t + C)]^{2/3} \;\Rightarrow$

$y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$

8. $\dfrac{dy}{d\theta} = \dfrac{e^y \sin^2\theta}{y\sec\theta} \;\Rightarrow\; \dfrac{y}{e^y}\,dy = \dfrac{\sin^2\theta}{\sec\theta}\,d\theta \;\Rightarrow\; \int ye^{-y}\,dy = \int \sin^2\theta\cos\theta\,d\theta$. Integrating the left side by parts with $u = y$, $dv = e^{-y}\,dy$ and the right side by the substitution $u = \sin\theta$, we obtain $-ye^{-y} - e^{-y} = \tfrac{1}{3}\sin^3\theta + C$. We cannot solve explicitly for y.

10. $\dfrac{dz}{dt} + e^{t+z} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -e^t e^z \;\Rightarrow\; \int e^{-z}\,dz = -\int e^t\,dt \;\Rightarrow\; -e^{-z} = -e^t + C \;\Rightarrow\; e^{-z} = e^t - C \;\Rightarrow$

$\dfrac{1}{e^z} = e^t - C \;\Rightarrow\; e^z = \dfrac{1}{e^t - C} \;\Rightarrow\; z = \ln\left(\dfrac{1}{e^t - C}\right) \;\Rightarrow\; z = -\ln\left(e^t - C\right)$

16. $xy' + y = y^2 \;\Rightarrow\; x\dfrac{dy}{dx} = y^2 - y \;\Rightarrow\; x\,dy = (y^2 - y)\,dx \;\Rightarrow\; \dfrac{dy}{y^2 - y} = \dfrac{dx}{x} \;\Rightarrow$

$\displaystyle\int \frac{dy}{y(y-1)} = \int \frac{dx}{x} \;\; [y \neq 0, 1] \;\Rightarrow\; \int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \int \frac{dx}{x} \;\Rightarrow\; \ln|y - 1| - \ln|y| = \ln|x| + C \;\Rightarrow$

$\ln\left|\dfrac{y-1}{y}\right| = \ln\left(e^C|x|\right) \;\Rightarrow\; \left|\dfrac{y-1}{y}\right| = e^C|x| \;\Rightarrow\; \dfrac{y-1}{y} = Kx$, where $K = \pm e^C \;\Rightarrow\; 1 - \dfrac{1}{y} = Kx \;\Rightarrow$

$\dfrac{1}{y} = 1 - Kx \;\Rightarrow\; y = \dfrac{1}{1 - Kx}$. [The excluded cases, $y = 0$ and $y = 1$, are ruled out by the initial condition $y(1) = -1$.]

Now $y(1) = -1 \;\Rightarrow\; -1 = \dfrac{1}{1-K} \;\Rightarrow\; 1 - K = -1 \;\Rightarrow\; K = 2$, so $y = \dfrac{1}{1 - 2x}$.

19. If the slope at the point (x, y) is xy, then we have $\dfrac{dy}{dx} = xy \;\Rightarrow\; \dfrac{dy}{y} = x\,dx \;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int x\,dx \;\Rightarrow$

$\ln|y| = \tfrac{1}{2}x^2 + C$. $y(0) = 1 \;\Rightarrow\; \ln 1 = 0 + C \;\Rightarrow\; C = 0$. Thus, $|y| = e^{x^2/2} \;\Rightarrow\; y = \pm e^{x^2/2}$, so $y = e^{x^2/2}$ since $y(0) = 1 > 0$. Note that $y = 0$ is not a solution because it doesn't satisfy the initial condition $y(0) = 1$.

43. Let $y(t)$ be the amount of alcohol in the vat after t minutes. Then $y(0) = 0.04(500) = 20$ gal. The amount of beer in the vat is 500 gallons at all times, so the percentage at time t (in minutes) is $y(t)/500 \times 100$, and the change in the amount of alcohol with respect to time t is $\frac{dy}{dt} = \text{rate in} - \text{rate out} = 0.06\left(5\,\frac{\text{gal}}{\text{min}}\right) - \frac{y(t)}{500}\left(5\,\frac{\text{gal}}{\text{min}}\right) = 0.3 - \frac{y}{100} = \frac{30-y}{100}\,\frac{\text{gal}}{\text{min}}$.

Hence, $\int \frac{dy}{30-y} = \int \frac{dt}{100}$ and $-\ln|30-y| = \frac{1}{100}t + C$. Because $y(0) = 20$, we have $-\ln 10 = C$, so $-\ln|30-y| = \frac{1}{100}t - \ln 10 \quad \Rightarrow \quad \ln|30-y| = -t/100 + \ln 10 \quad \Rightarrow \quad \ln|30-y| = \ln e^{-t/100} + \ln 10 \quad \Rightarrow$ $\ln|30-y| = \ln(10e^{-t/100}) \quad \Rightarrow \quad |30-y| = 10e^{-t/100}$. Since y is continuous, $y(0) = 20$, and the right-hand side is never zero, we deduce that $30 - y$ is always positive. Thus, $30 - y = 10e^{-t/100} \quad \Rightarrow \quad y = 30 - 10e^{-t/100}$. The percentage of alcohol is $p(t) = y(t)/500 \times 100 = y(t)/5 = 6 - 2e^{-t/100}$. The percentage of alcohol after one hour is $p(60) = 6 - 2e^{-60/100} \approx 4.9$.