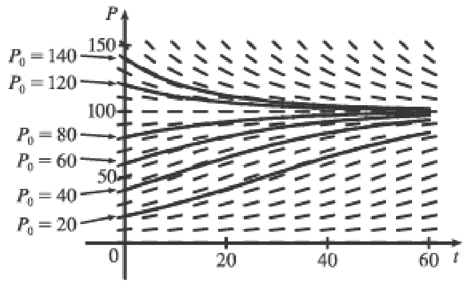
1. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100)$. Comparing to Equation 1, $dP/dt = kP(1 - P/K)$, we see that the carrying capacity is $K = 100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

(c)



All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

(d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \to \infty$.

3. (a) $\dfrac{dy}{dt} = ky\left(1 - \frac{y}{K}\right) \quad \Rightarrow \quad y(t) = \dfrac{K}{1 + Ae^{-kt}}$ with $A = \dfrac{K - y(0)}{y(0)}$. With $K = 8 \times 10^7$, $k = 0.71$, and $y(0) = 2 \times 10^7$, we get the model $y(t) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}}$, so $y(1) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 \quad \Rightarrow \quad \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \quad \Rightarrow \quad 2 = 1 + 3e^{-0.71t} \quad \Rightarrow \quad e^{-0.71t} = \frac{1}{3} \quad \Rightarrow$

$-0.71t = \ln \frac{1}{3} \quad \Rightarrow \quad t = \dfrac{\ln 3}{0.71} \approx 1.55$ years

7. (a) Our assumption is that $\frac{dy}{dt} = ky(1-y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (1), $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$, we substitute $y = \frac{P}{K}$, $P = Ky$, and $\frac{dP}{dt} = K\frac{dy}{dt}$,

to obtain $K\frac{dy}{dt} = k(Ky)(1-y) \quad \Leftrightarrow \quad \frac{dy}{dt} = ky(1-y)$, our equation in part (a).

Now the solution to (1) is $P(t) = \frac{K}{1 + Ae^{-kt}}$, where $A = \frac{K - P_0}{P_0}$.

We use the same substitution to obtain $Ky = \frac{K}{1 + \frac{K - Ky_0}{Ky_0}e^{-kt}} \quad \Rightarrow \quad y = \frac{y_0}{y_0 + (1-y_0)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in the solution of Equation 5.

(c) Let t be the number of hours since 8 AM. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so

$\frac{1}{2} = y(4) = \frac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$,

so $y = \frac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \frac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y + 23y\left(\frac{2}{23}\right)^{t/4} = 2 \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4} = \frac{2-2y}{23y} \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23}\cdot\frac{1-y}{y} \quad \Rightarrow \quad \left(\frac{2}{23}\right)^{t/4-1} = \frac{1-y}{y}.$$

It follows that $\frac{t}{4} - 1 = \frac{\ln[(1-y)/y]}{\ln\frac{2}{23}}$, so $t = 4\left[1 + \frac{\ln((1-y)/y)}{\ln\frac{2}{23}}\right]$.

When $y = 0.9$, $\frac{1-y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \frac{\ln 9}{\ln\frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

9. (a) $$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \quad \Rightarrow \quad \frac{d^2P}{dt^2} = k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right)\frac{dP}{dt}\right] = k\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right)$$
$$= k\left[kP\left(1 - \frac{P}{K}\right)\right]\left(1 - \frac{2P}{K}\right) = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \quad \Leftrightarrow \quad P = 0$, $P = K$, or $P = K/2$. Since $0 < P < K$, we see that $P'' = 0 \quad \Leftrightarrow \quad P = K/2$.