Calculus & Analytic Geometry II

The Chain Rule and Substitution

Warm-up. Find the following derivatives

$\left(e^{\sin x}\right)' =$ $\frac{d}{dx} \ln|\sec x| =$ $(\cos(\ln x))' =$

What integrals correspond to the problems above?

Analogy *derivative:integral::chain rule: substitution*

Recall for any differentiable function $u(x)$ and $n \neq 1$, the Chain Rule applied to the Power Rule says

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \cdot \frac{du}{dx}.$$

So $\int u^n \cdot \frac{du}{dx} dx =$

Let's try to apply this idea.

$\int (2x+3)^4 dx$ $\int \frac{ds}{\sqrt{5s-4}}$

The general form of the Chain Rule says $F(g(x))' = F'(g(x)) \cdot g'(x)$. So

$$\int F'(g(x)) \cdot g'(x)dx =$$

The Substitution Rule. *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

This allows us to replace a complicated integral by a (hopefully) simpler one. Let's see how this helps with our original problems of the day.

$$\int \cos x e^{\sin x} dx =$$

$$\int \tan x dx =$$

$$\int \frac{1}{x} \sin(\ln x) dx =$$

Practice Problems. Evaluate the following integrals.

1. $\displaystyle\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

2. $\displaystyle\int \frac{6\cos t}{(2+\sin t)^3}dt$

3. $\displaystyle\int \sec^2 x \tan x dx$

4. $\displaystyle\int \sec^2 x \tan^2 x dx$

5. $\displaystyle\int \sec^4 x \tan x dx$

6. $\displaystyle\int \frac{\ln\sqrt{t}}{t}dt$

The Substitution Rule for Definite Integrals. *If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof. Let F be an antiderivative of f. So
$\displaystyle\int_a^b f(g(x))g'(x)dx = F(g(x)) \mid_a^b =$

Revisiting Practice Problems. Evaluate the following definite integrals.

1. $\displaystyle\int_0^2 \frac{4ydy}{\sqrt{2y^2+1}}$

2. $\displaystyle\int_0^{\pi/2} \frac{6\cos t}{(2+\sin t)^3}dt$

3. $\displaystyle\int_{\pi/6}^{\pi/4} \sec^2 x \tan x dx$