Calculus & Analytic Geometry II

Separation of Variables

Big Idea. For nice differential equations, we can sometime find a simple, closed form solution.

Recall Example. Suppose $y' = y + 1$ and $y(0) = 1$. We estimated $y(1)$ using $\Delta t = \frac{1}{3}$ and found $y(1) \approx 3.741$. Creating a spreadsheet makes this computation more palatable.

Summary of data

dt	$y(t)$
1/3	3.741
.1	4.187
.01	4.410
.001	
.0001	

Verify that $y(t) = 2e^t - 1$ is a solution to the given initial value problem.

Consider the differentials that appear in Leibnitz notation as variables that can be manipulated. Can you isolate all the ys on one side and the ts on the other?

$$\frac{dy}{dt} = y + 1$$

You have just completed your first analytical solution using the method of *separation of variables.*

More Practice.

$$\begin{cases} y' &= -2y \\ y(0) &= 1 \end{cases} \qquad \begin{cases} \dfrac{dH}{dt} &= -k(H-20) \\ H(0) &= 30 \end{cases} \qquad \begin{cases} \dfrac{dP}{dt} &= 2P - 2Pt \\ P(0) &= 5 \end{cases}$$

$y = e^{-2t}$
$H = 20 + 10e^{-kt}$
$P = 5e^{2t-t^2}$

Question. What property makes these *nice* differential equations?

Justification for Separation of Variables

Suppose $\dfrac{dy}{dx} = g(x)f(y)$ and $f(y) \neq 0$. Let $h(y) = 1/f(y)$ and rewrite

Integrate both sides with respect to x:

$$\int h(y)\frac{dy}{dx}dx = \int g(x)dx$$

Substitute using
$u = y(x)$
$du =$

$$\int h(y)dy = \int g(x)dx$$

Another case where the Leibniz notation makes the chain rule look like cancellation.

Example. On page 42, we showed that $P(t) = \dfrac{1}{1+2e^{-kt}}$ was a one solution to the differential equation $\dfrac{dP}{dt} = kP(1-P)$. Can we now see why?

Problem. A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour? $(\approx 4.9\%)$

Revisit Example 9.1.1, pg. 570. Find the general solution to $y' = \frac{1}{2}(y^2 - 1)$.

$(1 \pm ce^x)/(1 \mp ce^x)$