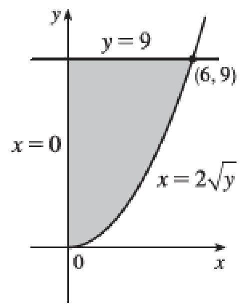
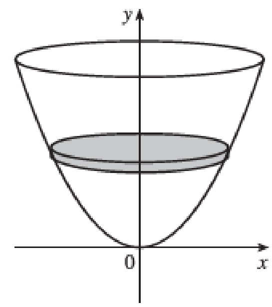
5. A cross-section is a disk with radius $2\sqrt{y}$, so its area is

$$A(y) = \pi\left(2\sqrt{y}\right)^2.$$

$$V = \int_0^9 A(y)\,dy = \int_0^9 \pi\left(2\sqrt{y}\right)^2 dy = 4\pi\int_0^9 y\,dy$$

$$= 4\pi\left[\tfrac{1}{2}y^2\right]_0^9 = 2\pi(81) = 162\pi$$





11. A cross-section is a washer with inner radius $1-\sqrt{x}$ and outer radius $1-x$, so its area is

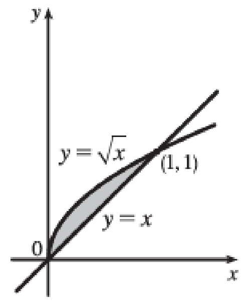
$$A(x) = \pi(1-x)^2 - \pi\left(1-\sqrt{x}\right)^2$$

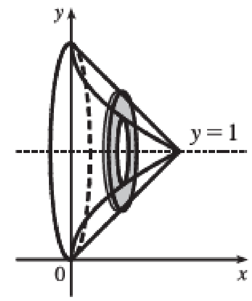
$$= \pi\left[(1-2x+x^2) - \left(1-2\sqrt{x}+x\right)\right]$$

$$= \pi\left(-3x+x^2+2\sqrt{x}\right).$$

$$V = \int_0^1 A(x)\,dx = \pi\int_0^1 \left(-3x+x^2+2\sqrt{x}\right) dx$$

$$= \pi\left[-\tfrac{3}{2}x^2+\tfrac{1}{3}x^3+\tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2}+\tfrac{5}{3}\right) = \tfrac{\pi}{6}$$



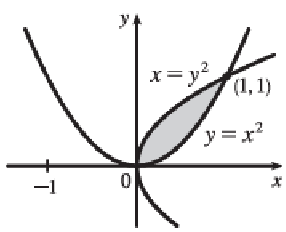


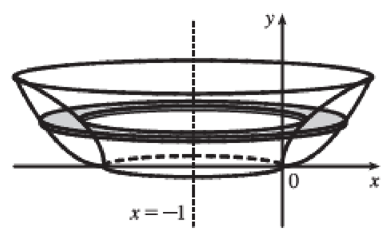
17. $y = x^2 \;\Rightarrow\; x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x=-1$ to $x=\sqrt{y}$ and the inner radius is the distance from $x=-1$ to $x=y^2$.

$$V = \int_0^1 \pi\left\{\left[\sqrt{y}-(-1)\right]^2 - \left[y^2-(-1)\right]^2\right\} dy = \pi\int_0^1 \left[\left(\sqrt{y}+1\right)^2 - (y^2+1)^2\right] dy$$

$$= \pi\int_0^1 \left(y+2\sqrt{y}+1-y^4-2y^2-1\right) dy = \pi\int_0^1 \left(y+2\sqrt{y}-y^4-2y^2\right) dy$$

$$= \pi\left[\tfrac{1}{2}y^2+\tfrac{4}{3}y^{3/2}-\tfrac{1}{5}y^5-\tfrac{2}{3}y^3\right]_0^1 = \pi\left(\tfrac{1}{2}+\tfrac{4}{3}-\tfrac{1}{5}-\tfrac{2}{3}\right) = \tfrac{29}{30}\pi$$





19. $\mathcal{R}_1$ about OA (the line $y=0$): $V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^3)^2\,dx = \pi\int_0^1 x^6\,dx = \pi\left[\frac{1}{7}x^7\right]_0^1 = \frac{\pi}{7}$

20. $\mathcal{R}_1$ about OC (the line $x=0$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1)^2 - \pi\left(\sqrt[3]{y}\right)^2\right] dy = \pi\int_0^1 (1-y^{2/3})\,dy = \pi\left[y-\tfrac{3}{5}y^{5/3}\right]_0^1 = \pi\left(1-\tfrac{3}{5}\right) = \tfrac{2}{5}\pi$$

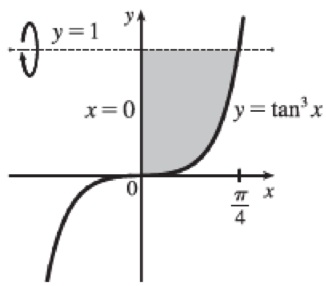
21. $\mathcal{R}_1$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left(1 - \sqrt[3]{y}\right)^2 dy = \pi\int_0^1 (1 - 2y^{1/3} + y^{2/3})\,dy = \pi\left[y - \tfrac{3}{2}y^{4/3} + \tfrac{3}{5}y^{5/3}\right]_0^1$$
$$= \pi\left(1 - \tfrac{3}{2} + \tfrac{3}{5}\right) = \tfrac{\pi}{10}$$

22. $\mathcal{R}_1$ about BC (the line $y = 1$):

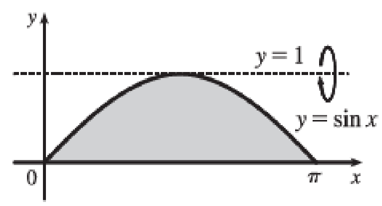
$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1)^2 - \pi(1 - x^3)^2\right] dx = \pi\int_0^1 \left[1 - (1 - 2x^3 + x^6)\right] dx$$
$$= \pi\int_0^1 (2x^3 - x^6)\,dx = \pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{7}\right) = \tfrac{5}{14}\pi$$

31. $V = \pi \displaystyle\int_0^{\pi/4} (1 - \tan^3 x)^2\,dx$



33. $V = \pi \displaystyle\int_0^{\pi} \left[(1 - 0)^2 - (1 - \sin x)^2\right] dx$

$= \pi \displaystyle\int_0^{\pi} \left[1^2 - (1 - \sin x)^2\right] dx$



47. (a) $V = \int_2^{10} \pi\,[f(x)]^2\,dx \approx \pi\frac{10-2}{4}\left\{[f(3)]^2 + [f(5)]^2 + [f(7)]^2 + [f(9)]^2\right\}$

$\approx 2\pi\left[(1.5)^2 + (2.2)^2 + (3.8)^2 + (3.1)^2\right] \approx 196 \text{ units}^3$

(b) $V = \int_0^4 \pi\left[(\text{outer radius})^2 - (\text{inner radius})^2\right] dy$

$\approx \pi\frac{4-0}{4}\left\{\left[(9.9)^2 - (2.2)^2\right] + \left[(9.7)^2 - (3.0)^2\right] + \left[(9.3)^2 - (5.6)^2\right] + \left[(8.7)^2 - (6.5)^2\right]\right\}$

$\approx 838 \text{ units}^3$

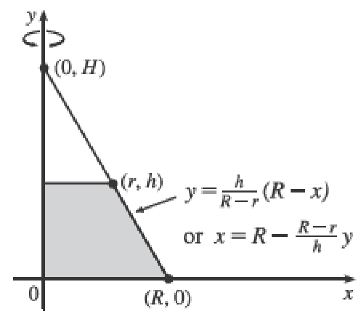
50. $V = \pi \int_0^h \left(R - \frac{R-r}{h}y\right)^2 dy$

$= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h}y + \left(\frac{R-r}{h}\right)^2 y^2\right] dy$

$= \pi \left[R^2 y - \frac{R(R-r)}{h}y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3\right]_0^h$

$= \pi \left[R^2 h - R(R-r)h + \frac{1}{3}(R-r)^2 h\right]$

$= \frac{1}{3}\pi h\left[3Rr + (R^2 - 2Rr + r^2)\right] = \frac{1}{3}\pi h(R^2 + Rr + r^2)$



Another solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore,

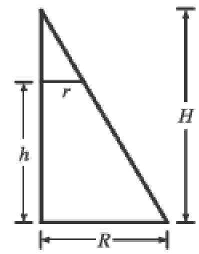
$Hr = HR - hR \;\Rightarrow\; hR = H(R-r) \;\Rightarrow\; H = \frac{hR}{R-r}$. Now

$V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 (H-h)$ [by Exercise 49]

$= \frac{1}{3}\pi R^2 \frac{hR}{R-r} - \frac{1}{3}\pi r^2 \frac{rh}{R-r}$ $\left[H - h = \frac{rH}{R} = \frac{rhR}{R(R-r)}\right]$

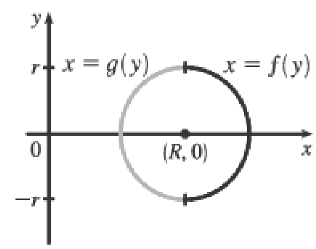
$= \frac{1}{3}\pi h \frac{R^3 - r^3}{R-r} = \frac{1}{3}\pi h(R^2 + Rr + r^2)$

$= \frac{1}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right]h = \frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)h$



where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 52 for a related result.)

63. (a) The torus is obtained by rotating the circle $(x-R)^2 + y^2 = r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by $x = R + \sqrt{r^2 - y^2} = f(y)$ and the left half by $x = R - \sqrt{r^2 - y^2} = g(y)$. So



$V = \pi \int_{-r}^{r} \{[f(y)]^2 - [g(y)]^2\}\, dy$

$= 2\pi \int_0^r \left[\left(R^2 + 2R\sqrt{r^2 - y^2} + r^2 - y^2\right) - \left(R^2 - 2R\sqrt{r^2 - y^2} + r^2 - y^2\right)\right] dy$

$= 2\pi \int_0^r 4R\sqrt{r^2 - y^2}\, dy = 8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy$

(b) Observe that the integral represents a quarter of the area of a circle with radius r, so

$8\pi R \int_0^r \sqrt{r^2 - y^2}\, dy = 8\pi R \cdot \frac{1}{4}\pi r^2 = 2\pi^2 r^2 R$.