5. The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by $\int_0^8 F(x)\,dx = \int_0^4 F(x)\,dx + \int_4^8 F(x)\,dx = \frac{1}{2}(4)(30) + (4)(30) = 180$ J.

7. $10 = f(x) = kx = \frac{1}{3}k$ [4 inches $= \frac{1}{3}$ foot], so $k = 30$ lb/ft and $f(x) = 30x$. Now 6 inches $= \frac{1}{2}$ foot, so $W = \int_0^{1/2} 30x\,dx = \left[15x^2\right]_0^{1/2} = \frac{15}{4}$ ft-lb.

13. (a) The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}x_i^*\,\Delta x = \int_0^{50} \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25} \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft-lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50} \frac{1}{2}\cdot 25\,dx = \frac{25}{2}\left[x\right]_{25}^{50} = \frac{625}{2}$ ft-lb. The total work done in pulling half the rope to the top of the building is $W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4}\cdot 625 = \frac{1875}{4}$ ft-lb.

15. The work needed to lift the cable is $\lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^*\,\Delta x = \int_0^{500} 2x\,dx = \left[x^2\right]_0^{500} = 250{,}000$ ft-lb. The work needed to lift the coal is 800 lb $\cdot$ 500 ft $= 400{,}000$ ft-lb. Thus, the total work required is $250{,}000 + 400{,}000 = 650{,}000$ ft-lb.

21. A rectangular "slice" of water Δx m thick and lying x m above the bottom has width x m and volume $8x\,\Delta x$ m^3. It weighs about $(9.8\times 1000)(8x\,\Delta x)$ N, and must be lifted $(5-x)$ m by the pump, so the work needed is about $(9.8\times 10^3)(5-x)(8x\,\Delta x)$ J. The total work required is

$$\begin{aligned} W &\approx \int_0^3 (9.8\times 10^3)(5-x)8x\,dx = (9.8\times 10^3)\int_0^3 (40x - 8x^2)\,dx = (9.8\times 10^3)\left[20x^2 - \tfrac{8}{3}x^3\right]_0^3 \\ &= (9.8\times 10^3)(180-72) = (9.8\times 10^3)(108) = 1058.4\times 10^3 \approx 1.06\times 10^6 \text{ J} \end{aligned}$$

22. Let y measure depth (in meters) below the center of the spherical tank, so that $y = -3$ at the top of the tank and $y = -4$ at the spigot. A horizontal disk-shaped "slice" of water Δy m thick and lying at coordinate y has radius $\sqrt{9-y^2}$ m and volume $\pi r^2\Delta y = \pi(9-y^2)\,\Delta y$ m^3. It weighs about $(9.8\times 1000)\pi(9-y^2)\,\Delta y$ N must be lifted $(y+4)$ m by the pump, so the work needed to pump it out is about $(9.8\times 10^3)(y+4)\pi(9-y^2)\,\Delta y$ J. The total work required is

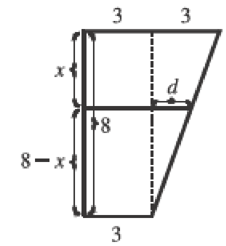
$$\begin{aligned} W &\approx \int_{-3}^{3} (9.8\times 10^3)(y+4)\pi(9-y^2)\,dy = (9.8\times 10^3)\pi\int_{-3}^{3}(9y - y^3 + 36 - 4y^2)\,dy \\ &= (9.8\times 10^3)\pi(2)(4)\int_0^3 (9-y^2)\,dy \qquad \text{[by Theorem 5.5.7]} \\ &= (78.4\times 10^3)\pi\left[9y - \tfrac{1}{3}y^3\right]_0^3 = (78.4\times 10^3)\pi(18) = 1{,}411{,}200\pi \approx 4.43\times 10^6 \text{ J} \end{aligned}$$

23. Let x measure depth (in feet) below the spout at the top of the tank. A horizontal disk-shaped "slice" of water Δx ft thick and lying at coordinate x has radius $\frac{3}{8}(16-x)$ ft $(\star)$ and volume $\pi r^2 \Delta x = \pi \cdot \frac{9}{64}(16-x)^2\, \Delta x$ ft^3. It weighs about $(62.5)\frac{9\pi}{64}(16-x)^2\, \Delta x$ lb and must be lifted x ft by the pump, so the work needed to pump it out is about $(62.5)x\, \frac{9\pi}{64}(16-x)^2\, \Delta x$ ft-lb. The total work required is

$$W \approx \int_0^8 (62.5)x\, \tfrac{9\pi}{64}(16-x)^2\, dx = (62.5)\tfrac{9\pi}{64}\int_0^8 x(256-32x+x^2)\, dx$$

$$= (62.5)\tfrac{9\pi}{64}\int_0^8 (256x-32x^2+x^3)\, dx = (62.5)\tfrac{9\pi}{64}\left[128x^2 - \tfrac{32}{3}x^3 + \tfrac{1}{4}x^4\right]_0^8$$

$$= (62.5)\frac{9\pi}{64}\left(\frac{11,264}{3}\right) = 33{,}000\pi \approx 1.04 \times 10^5 \text{ ft-lb}$$



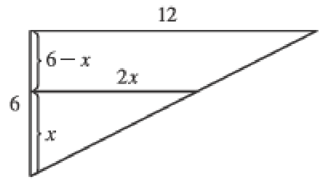
$(\star)$ From similar triangles, $\dfrac{d}{8-x} = \dfrac{3}{8}$.

So $r = 3 + d = 3 + \frac{3}{8}(8-x)$

$= \dfrac{3(8)}{8} + \dfrac{3}{8}(8-x)$

$= \frac{3}{8}(16-x)$

24. Let x measure the distance (in feet) above the bottom of the tank. A horizontal "slice" of water Δx ft thick and lying at coordinate x has volume $10(2x)\,\Delta x$ ft^3. It weights about $(62.5)20x\,\Delta x$ lb and must be lifted $(6-x)$ ft by the pump, so the work needed to pump it out is about $(62.5)(6-x)20x\,\Delta x$ ft-lb. The total work required is



$$W \approx \int_0^6 (62.5)(6-x)20x\, dx = 1250\int_0^6 (6x-x^2)\, dx = 1250\left[3x^2 - \tfrac{1}{3}x^3\right]_0^6 = 1250(36) = 45{,}000 \text{ ft-lb.}$$