5. Let $t = \sec\theta$, so $dt = \sec\theta\,\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\,\tan\theta}\sec\theta\,\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

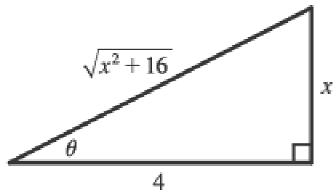
$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+16} = \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta+1)}$$

$$= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|$$

$= 4\sec\theta$ for the relevant values of θ.



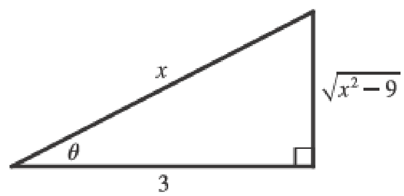
$$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1$$

$$= \ln\left|\sqrt{x^2+16} + x\right| - \ln|4| + C_1 = \ln\left(\sqrt{x^2+16}+x\right) + C, \text{ where } C = C_1 - \ln 4.$$

(Since $\sqrt{x^2+16} + x > 0$, we don't need the absolute value.)

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

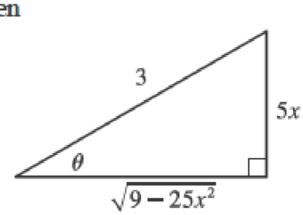
17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C.$

21. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.6 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

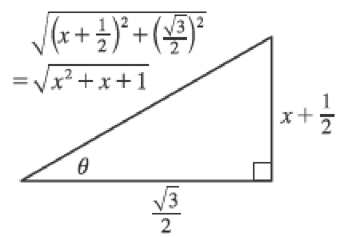
$$\int_0^{0.6}\frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{\pi/2}\frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\frac{3}{5}\cos\theta\,d\theta\right) = \frac{9}{125}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \tfrac{9}{125}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{9}{250}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \tfrac{9}{250}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \tfrac{9}{500}\pi$$



25. $x^2 + x + 1 = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$. Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, so $dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$ and $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\sec\theta$.



Then

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}}{\frac{\sqrt{3}}{2}\sec\theta}\,\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$$

$$= \int \left(\tfrac{\sqrt{3}}{2}\tan\theta - \tfrac{1}{2}\right)\sec\theta\, d\theta = \int \tfrac{\sqrt{3}}{2}\tan\theta\,\sec\theta\, d\theta - \int \tfrac{1}{2}\sec\theta\, d\theta$$

$$= \tfrac{\sqrt{3}}{2}\sec\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C_1$$

$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\sqrt{x^2+x+1} + \tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right| + C_1$$

$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\left[\sqrt{x^2+x+1} + \left(x+\tfrac{1}{2}\right)\right]\right| + C_1$$

$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C_1$$

$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C, \quad \text{where } C = C_1 - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}}$$

29. Let $u = x^2$, $du = 2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx = \int \sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \cos\theta \cdot \cos\theta\, d\theta \qquad \left[\begin{array}{c}\text{where } u = \sin\theta,\ du = \cos\theta\, d\theta,\\ \text{and } \sqrt{1-u^2} = \cos\theta\end{array}\right]$$

$$= \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)\, d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\,\cos\theta + C$$

$$= \tfrac{1}{4}\sin^{-1}u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C$$

33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\, d\theta \qquad \left[\begin{array}{c}\text{where } x = \sec\theta,\ dx = \sec\theta\tan\theta\, d\theta,\\ \sqrt{x^2-1} = \tan\theta, \text{ and } \alpha = \sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\, d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\, d\theta = \tfrac{1}{6}\left[\tan\theta - \theta\right]_0^\alpha$$

$$= \tfrac{1}{6}(\tan\alpha - \alpha) = \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$

34. $9x^2 - 4y^2 = 36 \quad \Rightarrow \quad y = \pm\frac{3}{2}\sqrt{x^2-4} \quad \Rightarrow$

area $= 2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx$

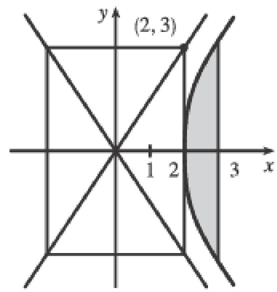
$= 3\int_0^\alpha 2\tan\theta\, 2\sec\theta\, \tan\theta\, d\theta \quad \begin{bmatrix} \text{where } x = 2\sec\theta, \\ dx = 2\sec\theta\, \tan\theta\, d\theta, \\ \alpha = \sec^{-1}\left(\frac{3}{2}\right) \end{bmatrix}$

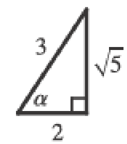
$= 12\int_0^\alpha \left(\sec^2\theta - 1\right)\sec\theta\, d\theta = 12\int_0^\alpha \left(\sec^3\theta - \sec\theta\right) d\theta$

$= 12\left[\frac{1}{2}(\sec\theta\, \tan\theta + \ln|\sec\theta + \tan\theta|) - \ln|\sec\theta + \tan\theta|\right]_0^\alpha$

$= 6\left[\sec\theta\, \tan\theta - \ln|\sec\theta + \tan\theta|\right]_0^\alpha$

$= 6\left[\frac{3\sqrt{5}}{4} - \ln\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)\right] = \frac{9\sqrt{5}}{2} - 6\ln\left(\frac{3+\sqrt{5}}{2}\right)$





40. The curves intersect when $x^2 + \left(\frac{1}{2}x^2\right)^2 = 8 \quad \Leftrightarrow \quad x^2 + \frac{1}{4}x^4 = 8 \quad \Leftrightarrow \quad x^4 + 4x^2 - 32 = 0 \quad \Leftrightarrow$

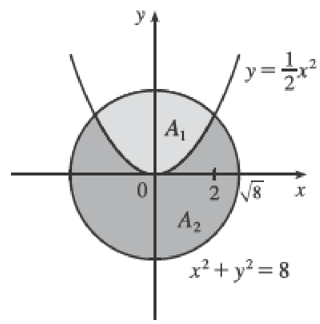
$(x^2+8)(x^2-4) = 0 \quad \Leftrightarrow \quad x = \pm 2$. The area inside the circle and above the parabola is given by

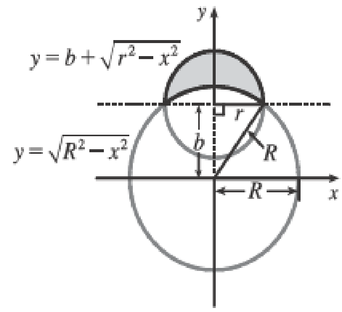
$A_1 = \int_{-2}^2 \left(\sqrt{8-x^2} - \frac{1}{2}x^2\right) dx = 2\int_0^2 \sqrt{8-x^2}\,dx - 2\int_0^2 \frac{1}{2}x^2\,dx$

$= 2\left[\frac{1}{2}(8)\sin^{-1}\left(\frac{2}{\sqrt{8}}\right) + \frac{1}{2}(2)\sqrt{8-2^2} - \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2\right]$ [by Exercise 39]

$= 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + 2\sqrt{4} - \frac{8}{3} = 8\left(\frac{\pi}{4}\right) + 4 - \frac{8}{3} = 2\pi + \frac{4}{3}$

Since the area of the disk is $\pi\left(\sqrt{8}\right)^2 = 8\pi$, the area inside the circle and below the parabola ia $A_2 = 8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3}$.





41. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is

$$A = \int_{-r}^{r} \left[\left(b + \sqrt{r^2 - x^2} \right) - \sqrt{R^2 - x^2} \right] dx$$
$$= 2 \int_0^r \left(b + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2} \right) dx$$
$$= 2 \int_0^r b\, dx + 2 \int_0^r \sqrt{r^2 - x^2}\, dx - 2 \int_0^r \sqrt{R^2 - x^2}\, dx$$

The first integral is just $2br = 2r\sqrt{R^2 - r^2}$. The second integral represents the area of a quarter-circle of radius r, so its value is $\frac{1}{4}\pi r^2$. To evaluate the other integral, note that

$$\int \sqrt{a^2 - x^2}\, dx = \int a^2 \cos^2\theta\, d\theta \quad [x = a\sin\theta,\ dx = a\cos\theta\, d\theta] \ = \left(\tfrac{1}{2}a^2\right) \int (1 + \cos 2\theta)\, d\theta$$
$$= \tfrac{1}{2}a^2 \left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2 (\theta + \sin\theta\ \cos\theta) + C$$
$$= \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2 - x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

Thus, the desired area is

$$A = 2r\sqrt{R^2 - r^2} + 2\left(\tfrac{1}{4}\pi r^2\right) - \left[R^2 \arcsin(x/R) + x\sqrt{R^2 - x^2}\right]_0^r$$
$$= 2r\sqrt{R^2 - r^2} + \tfrac{1}{2}\pi r^2 - \left[R^2 \arcsin(r/R) + r\sqrt{R^2 - r^2}\right] = r\sqrt{R^2 - r^2} + \tfrac{\pi}{2} r^2 - R^2 \arcsin(r/R)$$