2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1, 2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

7. $$\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty} \int_t^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty} \left[-2\sqrt{2-w}\right]_t^{-1} \qquad [u = 2-w,\ du = -dw]$$
$$= \lim_{t\to-\infty} \left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty. \qquad \text{Divergent}$$

8. $$\int_0^{\infty} \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right)$$
$$= \tfrac{1}{2}\left(0 + \tfrac{1}{2}\right) = \tfrac{1}{4}. \qquad \text{Convergent}$$

11. $$\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2} = \int_{-\infty}^{0} \frac{x\,dx}{1+x^2} + \int_0^{\infty} \frac{x\,dx}{1+x^2} \text{ and}$$
$$\int_{-\infty}^{0} \frac{x\,dx}{1+x^2} = \lim_{t\to-\infty} \left[\tfrac{1}{2}\ln(1+x^2)\right]_t^0 = \lim_{t\to-\infty}\left[0 - \tfrac{1}{2}\ln(1+t^2)\right] = -\infty. \qquad \text{Divergent}$$

14. $$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^{\sqrt{t}} e^{-u}(2\,du) \qquad \begin{bmatrix} u = \sqrt{x}, \\ du = dx/\left(2\sqrt{x}\right) \end{bmatrix}$$
$$= 2\lim_{t\to\infty}\left[-e^{-u}\right]_1^{\sqrt{t}} = 2\lim_{t\to\infty}\left(-e^{-\sqrt{t}} + e^{-1}\right) = 2(0 + e^{-1}) = 2e^{-1}. \qquad \text{Convergent}$$

15. $\int_{2\pi}^{\infty} \sin\theta\,d\theta = \lim_{t\to\infty} \int_{2\pi}^t \sin\theta\,d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = \lim_{t\to\infty}(-\cos t + 1)$. This limit does not exist, so the integral is divergent. Divergent

17. $$\int_1^{\infty} \frac{x+1}{x^2+2x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x}\,dx = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t = \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+2t) - \ln 3\right] = \infty.$$
Divergent

25. $$\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} \int_1^{\ln t} u^{-3}\,du \qquad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} \qquad = \lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln t}$$
$$= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2} + \frac{1}{2}\right] = 0 + \frac{1}{2} = \frac{1}{2}. \qquad \text{Convergent}$$

27. $$\int_0^1 \frac{3}{x^5}\,dx = \lim_{t\to0^+} \int_t^1 3x^{-5}\,dx = \lim_{t\to0^+}\left[-\frac{3}{4x^4}\right]_t^1 = -\frac{3}{4}\lim_{t\to0^+}\left(1 - \frac{1}{t^4}\right) = \infty. \qquad \text{Divergent}$$

49. For $x > 0$, $\dfrac{x}{x^3+1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{x^3+1}\,dx$ is convergent by the Comparison Theorem. $\displaystyle\int_0^1 \frac{x}{x^3+1}\,dx$ is a constant, so $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx$ is also convergent.

50. For $x \geq 1$, $\dfrac{2+e^{-x}}{x} > \dfrac{2}{x}$ [since $e^{-x} > 0$] $> \dfrac{1}{x}$. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent by Equation 2 with $p = 1 \leq 1$, so $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

51. For $x > 1$, $f(x) = \dfrac{x+1}{\sqrt{x^4-x}} > \dfrac{x+1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, so $\displaystyle\int_2^\infty f(x)\,dx$ diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$, which diverges by Equation 2 with $p = 1 \leq 1$. Thus, $\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx$ also diverges.

57. If $p = 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} [\ln x]_t^1 = \infty$. Divergent.

If $p \neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p}$ [note that the integral is not improper if $p < 0$]

$$= \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\dfrac{1}{1-p}$.