

21. The moment M of the system about the origin is $M = \sum_{i=1}^2 m_i x_i = m_1 x_1 + m_2 x_2 = 40 \cdot 2 + 30 \cdot 5 = 230$.

The mass m of the system is $m = \sum_{i=1}^2 m_i = m_1 + m_2 = 40 + 30 = 70$.

The center of mass of the system is $M/m = \frac{230}{70} = \frac{23}{7}$.

22. $M = m_1 x_1 + m_2 x_2 + m_3 x_3 = 25(-2) + 20(3) + 10(7) = 80$; $\bar{x} = M/(m_1 + m_2 + m_3) = \frac{80}{55} = \frac{16}{11}$.

25. Since the region in the figure is symmetric about the y -axis, we know

that $\bar{x} = 0$. The region is “bottom-heavy,” so we know that $\bar{y} < 2$,

and we might guess that $\bar{y} = 1.5$.

$$A = \int_{-2}^2 (4 - x^2) dx = 2 \int_0^2 (4 - x^2) dx = 2 \left[4x - \frac{1}{3}x^3 \right]_0^2$$

$$= 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3}.$$

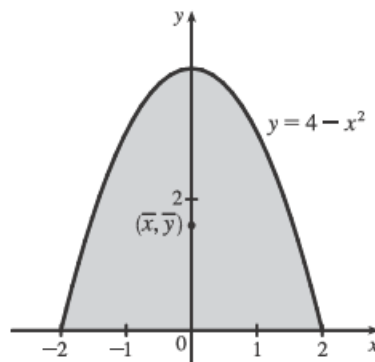
$$\bar{x} = \frac{1}{A} \int_{-2}^2 x(4 - x^2) dx = 0 \text{ since } f(x) = x(4 - x^2) \text{ is an odd}$$

function (or since the region is symmetric about the y -axis).

$$\bar{y} = \frac{1}{A} \int_{-2}^2 \frac{1}{2}(4 - x^2)^2 dx = \frac{3}{32} \cdot \frac{1}{2} \cdot 2 \int_0^2 (16 - 8x^2 + x^4) dx = \frac{3}{32} \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2$$

$$= \frac{3}{32} \left(32 - \frac{64}{3} + \frac{32}{5} \right) = 3 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = 3 \left(\frac{8}{15} \right) = \frac{8}{5}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(0, \frac{8}{5} \right)$.



27. The region in the figure is “right-heavy” and “bottom-heavy,” so we know

$\bar{x} > 0.5$ and $\bar{y} < 1$, and we might guess that $\bar{x} = 0.6$ and $\bar{y} = 0.9$.

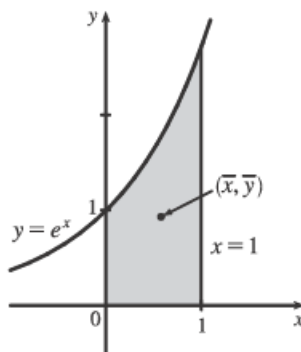
$$A = \int_0^1 e^x dx = [e^x]_0^1 = e - 1.$$

$$\bar{x} = \frac{1}{A} \int_0^1 x e^x dx = \frac{1}{e-1} [x e^x - e^x]_0^1 \quad \text{[by parts]}$$

$$= \frac{1}{e-1} [0 - (-1)] = \frac{1}{e-1}.$$

$$\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2}(e^x)^2 dx = \frac{1}{e-1} \cdot \frac{1}{4} [e^{2x}]_0^1 = \frac{1}{4(e-1)} (e^2 - 1) = \frac{e+1}{4}.$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4} \right) \approx (0.58, 0.93)$.

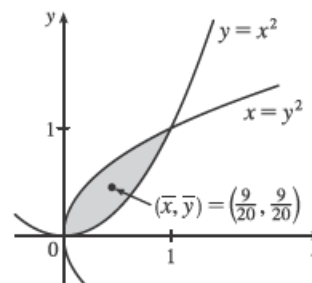


$$29. A = \int_0^1 (x^{1/2} - x^2) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right]_0^1 = \left(\frac{2}{3} - \frac{1}{3} \right) - 0 = \frac{1}{3}.$$

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^1 x(x^{1/2} - x^2) dx = 3 \int_0^1 (x^{3/2} - x^3) dx \\ &= 3 \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 \right]_0^1 = 3 \left(\frac{2}{5} - \frac{1}{4} \right) = 3 \left(\frac{3}{20} \right) = \frac{9}{20}. \end{aligned}$$

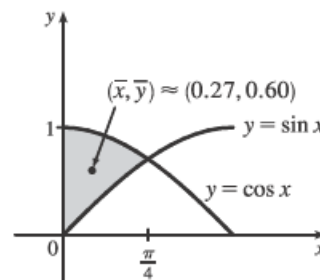
$$\begin{aligned} \bar{y} &= \frac{1}{A} \int_0^1 \frac{1}{2} \left[(x^{1/2})^2 - (x^2)^2 \right] dx = 3 \left(\frac{1}{2} \right) \int_0^1 (x - x^4) dx \\ &= \frac{3}{2} \left[\frac{1}{2}x^2 - \frac{1}{5}x^5 \right]_0^1 = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3}{2} \left(\frac{3}{10} \right) = \frac{9}{20}. \end{aligned}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{9}{20}, \frac{9}{20} \right)$.



$$31. A = \int_0^{\pi/4} (\cos x - \sin x) dx = [\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1.$$

$$\begin{aligned} \bar{x} &= A^{-1} \int_0^{\pi/4} x(\cos x - \sin x) dx \\ &= A^{-1} [x(\sin x + \cos x) + \cos x - \sin x]_0^{\pi/4} \quad \text{[integration by parts]} \\ &= A^{-1} \left(\frac{\pi}{4} \sqrt{2} - 1 \right) = \frac{\frac{1}{4} \pi \sqrt{2} - 1}{\sqrt{2} - 1}. \end{aligned}$$



$$\bar{y} = A^{-1} \int_0^{\pi/4} \frac{1}{2} (\cos^2 x - \sin^2 x) dx = \frac{1}{2A} \int_0^{\pi/4} \cos 2x dx = \frac{1}{4A} [\sin 2x]_0^{\pi/4} = \frac{1}{4A} = \frac{1}{4(\sqrt{2} - 1)}.$$

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{\pi \sqrt{2} - 4}{4(\sqrt{2} - 1)}, \frac{1}{4(\sqrt{2} - 1)} \right) \approx (0.27, 0.60)$.

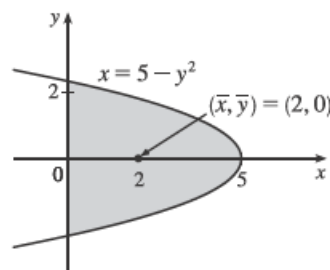
33. From the figure we see that $\bar{y} = 0$. Now

$$A = \int_0^5 2\sqrt{5-x} dx = 2 \left[-\frac{2}{3}(5-x)^{3/2} \right]_0^5 = 2 \left(0 + \frac{2}{3} \cdot 5^{3/2} \right) = \frac{20}{3} \sqrt{5},$$

so

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^5 x[\sqrt{5-x} - (-\sqrt{5-x})] dx = \frac{1}{A} \int_0^5 2x\sqrt{5-x} dx \\ &= \frac{1}{A} \int_{\sqrt{5}}^0 2(5-u^2)u(-2u) du \quad \left[\begin{array}{l} u = \sqrt{5-x}, x = 5-u^2, \\ u^2 = 5-x, dx = -2u du \end{array} \right] \\ &= \frac{4}{A} \int_0^{\sqrt{5}} u^2(5-u^2) du = \frac{4}{A} \left[\frac{5}{3}u^3 - \frac{1}{5}u^5 \right]_0^{\sqrt{5}} = \frac{3}{5\sqrt{5}} \left(\frac{25}{3} \sqrt{5} - 5\sqrt{5} \right) = 5 - 3 = 2. \end{aligned}$$

Thus, the centroid is $(\bar{x}, \bar{y}) = (2, 0)$.



40. Divide the lamina into three rectangles with masses 2, 2 and 6, with centroids $(-\frac{3}{2}, 1)$, $(0, \frac{1}{2})$ and $(2, \frac{3}{2})$, respectively.

The total mass of the lamina is 10. So, using Formulas 5, 6, and 7, we have

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \sum_{i=1}^3 m_i x_i = \frac{1}{10} [2(-\frac{3}{2}) + 2(0) + 6(2)] = \frac{1}{10}(9)$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \sum_{i=1}^3 m_i y_i = \frac{1}{10} [2(1) + 2(\frac{1}{2}) + 6(\frac{3}{2})] = \frac{1}{10}(12).$$

Thus, the centroid is $(\bar{x}, \bar{y}) = (\frac{9}{10}, \frac{6}{5})$.

41. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 39, the triangles have centroids $(-1, \frac{2}{3})$ and $(1, \frac{2}{3})$. The centroid of the rectangle (its center) is $(0, -\frac{1}{2})$.

So, using Formulas 5 and 7, we have $\bar{y} = \frac{M_x}{m} = \frac{1}{m} \sum_{i=1}^3 m_i y_i = \frac{1}{8} [2(\frac{2}{3}) + 2(\frac{2}{3}) + 4(-\frac{1}{2})] = \frac{1}{8}(\frac{2}{3}) = \frac{1}{12}$, and $\bar{x} = 0$,

since the lamina is symmetric about the line $x = 0$. Thus, the centroid is $(\bar{x}, \bar{y}) = (0, \frac{1}{12})$.