1. $y = x - x^{-1} \quad \Rightarrow \quad y' = 1 + x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\text{LHS}= xy' + y = x(1 + x^{-2}) + (x - x^{-1}) = x + x^{-1} + x - x^{-1} = 2x = \text{RHS}$$

2. $y = \sin x \, \cos x - \cos x \quad \Rightarrow \quad y' = \sin x\,(-\sin x) + \cos x\,(\cos x) - (-\sin x) = \cos^2 x - \sin^2 x + \sin x$.

$$\begin{aligned}\text{LHS} &= y' + (\tan x)y = \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x \, \cos x - \cos x)\\ &= \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x = \text{ RHS},\end{aligned}$$

so y is a solution of the differential equation. Also, $y(0) = \sin 0 \, \cos 0 - \cos 0 = 0 \cdot 1 - 1 = -1$, so the initial condition is satisfied.

3. (a) $y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2 e^{rx}$. Substituting these expressions into the differential equation $2y'' + y' - y = 0$, we get $2r^2 e^{rx} + re^{rx} - e^{rx} = 0 \quad \Rightarrow \quad (2r^2 + r - 1)e^{rx} = 0 \quad \Rightarrow$ $(2r - 1)(r + 1) = 0$ [since e^{rx} is never zero] $\Rightarrow \quad r = \frac{1}{2}$ or -1.

(b) Let $r_1 = \frac{1}{2}$ and $r_2 = -1$, so we need to show that every member of the family of functions $y = ae^{x/2} + be^{-x}$ is a solution of the differential equation $2y'' + y' - y = 0$.

$y = ae^{x/2} + be^{-x} \quad \Rightarrow \quad y' = \frac{1}{2}ae^{x/2} - be^{-x} \quad \Rightarrow \quad y'' = \frac{1}{4}ae^{x/2} + be^{-x}$.

$$\begin{aligned}\text{LHS} = 2y'' + y' - y &= 2\left(\tfrac{1}{4}ae^{x/2} + be^{-x}\right) + \left(\tfrac{1}{2}ae^{x/2} - be^{-x}\right) - (ae^{x/2} + be^{-x})\\ &= \tfrac{1}{2}ae^{x/2} + 2be^{-x} + \tfrac{1}{2}ae^{x/2} - be^{-x} - ae^{x/2} - be^{-x}\\ &= \left(\tfrac{1}{2}a + \tfrac{1}{2}a - a\right)e^{x/2} + (2b - b - b)e^{-x}\\ &= 0 = \text{RHS}\end{aligned}$$

5. (a) $y = \sin x \quad \Rightarrow \quad y' = \cos x \quad \Rightarrow \quad y'' = -\sin x$.

LHS $= y'' + y = -\sin x + \sin x = 0 \neq \sin x$, so $y = \sin x$ **is not** a solution of the differential equation.

(b) $y = \cos x \quad \Rightarrow \quad y' = -\sin x \quad \Rightarrow \quad y'' = -\cos x$.

LHS $= y'' + y = -\cos x + \cos x = 0 \neq \sin x$, so $y = \cos x$ **is not** a solution of the differential equation.

(c) $y = \frac{1}{2}x \sin x \quad \Rightarrow \quad y' = \frac{1}{2}(x \cos x + \sin x) \quad \Rightarrow \quad y'' = \frac{1}{2}(-x \sin x + \cos x + \cos x)$.

LHS $= y'' + y = \frac{1}{2}(-x \sin x + 2 \cos x) + \frac{1}{2}x \sin x = \cos x \neq \sin x$, so $y = \frac{1}{2}x \sin x$ **is not** a solution of the differential equation.

(d) $y = -\frac{1}{2}x \cos x \quad \Rightarrow \quad y' = -\frac{1}{2}(-x \sin x + \cos x) \quad \Rightarrow \quad y'' = -\frac{1}{2}(-x \cos x - \sin x - \sin x)$.

LHS $= y'' + y = -\frac{1}{2}(-x \cos x - 2 \sin x) + \left(-\frac{1}{2}x \cos x\right) = \sin x =$ RHS, so $y = -\frac{1}{2}x \cos x$ **is** a solution of the differential equation.

9. (a) $\dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right)$. Now $\dfrac{dP}{dt} > 0 \Rightarrow 1 - \dfrac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow \dfrac{P}{4200} < 1 \Rightarrow$ $P < 4200 \Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\dfrac{dP}{dt} < 0 \Rightarrow P > 4200$

(c) $\dfrac{dP}{dt} = 0 \Rightarrow P = 4200$ or $P = 0$

13. (a) P increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) $\dfrac{dP}{dt} = k(M - P)$ is always positive, so the level of performance P is increasing. As P gets close to M, dP/dt gets close to 0; that is, the performance levels off, as explained in part (a).

(c)