1. $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$ $[y \neq 0]$ $\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + C \Rightarrow$

$|y| = e^{\ln|x|+C} = e^{\ln|x|}e^C = e^C|x| \Rightarrow y = Kx$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

2. $\frac{dy}{dx} = \frac{\sqrt{x}}{e^y} \Rightarrow e^y\,dy = \sqrt{x}\,dx \Rightarrow \int e^y\,dy = \int x^{1/2}\,dx \Rightarrow e^y = \frac{2}{3}x^{3/2} + C \Rightarrow y = \ln\left(\frac{2}{3}x^{3/2} + C\right)$

3. $(x^2+1)y' = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{x\,dx}{x^2+1}$ $[y \neq 0]$ $\Rightarrow \int \frac{dy}{y} = \int \frac{x\,dx}{x^2+1} \Rightarrow$

$\ln|y| = \frac{1}{2}\ln(x^2+1) + C$ $[u = x^2+1, du = 2x\,dx]$ $= \ln(x^2+1)^{1/2} + \ln e^C = \ln\left(e^C\sqrt{x^2+1}\right) \Rightarrow$

$|y| = e^C\sqrt{x^2+1} \Rightarrow y = K\sqrt{x^2+1}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

4. $y' = y^2\sin x \Rightarrow \frac{dy}{dx} = y^2\sin x \Rightarrow \frac{dy}{y^2} = \sin x\,dx$ $[y \neq 0]$ $\Rightarrow \int \frac{dy}{y^2} = \int \sin x\,dx \Rightarrow$

$-\frac{1}{y} = -\cos x + C \Rightarrow \frac{1}{y} = \cos x - C \Rightarrow y = \frac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.

5. $(1+\tan y)\,y' = x^2+1 \Rightarrow (1+\tan y)\frac{dy}{dx} = x^2+1 \Rightarrow \left(1 + \frac{\sin y}{\cos y}\right)dy = (x^2+1)\,dx \Rightarrow$

$\int\left(1 - \frac{-\sin y}{\cos y}\right)dy = \int(x^2+1)\,dx \Rightarrow y - \ln|\cos y| = \frac{1}{3}x^3 + x + C.$

Note: The left side is equivalent to $y + \ln|\sec y|$.

13. $x\cos x = (2y + e^{3y})\,y' \Rightarrow x\cos x\,dx = (2y+e^{3y})\,dy \Rightarrow \int(2y+e^{3y})\,dy = \int x\cos x\,dx \Rightarrow$

$y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x + C$ [where the second integral is evaluated using integration by parts].

Now $y(0) = 0 \Rightarrow 0 + \frac{1}{3} = 0 + 1 + C \Rightarrow C = -\frac{2}{3}$. Thus, a solution is $y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x - \frac{2}{3}$.

We cannot solve explicitly for y.

19. If the slope at the point (x, y) is xy, then we have $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx$ $[y \neq 0]$ $\Rightarrow \int \frac{dy}{y} = \int x\,dx \Rightarrow$

$\ln|y| = \frac{1}{2}x^2 + C$. $y(0) = 1 \Rightarrow \ln 1 = 0 + C \Rightarrow C = 0$. Thus, $|y| = e^{x^2/2} \Rightarrow y = \pm e^{x^2/2}$, so $y = e^{x^2/2}$ since $y(0) = 1 > 0$. Note that $y = 0$ is not a solution because it doesn't satisfy the initial condition $y(0) = 1$.

39. (a) $\frac{dC}{dt} = r - kC \Rightarrow \frac{dC}{dt} = -(kC - r) \Rightarrow \int \frac{dC}{kC - r} = \int -dt \Rightarrow (1/k)\ \ln|kC - r| = -t + M_1 \Rightarrow$

$\ln|kC - r| = -kt + M_2 \Rightarrow |kC - r| = e^{-kt+M_2} \Rightarrow kC - r = M_3 e^{-kt} \Rightarrow kC = M_3 e^{-kt} + r \Rightarrow$

$C(t) = M_4 e^{-kt} + r/k$. $C(0) = C_0 \Rightarrow C_0 = M_4 + r/k \Rightarrow M_4 = C_0 - r/k \Rightarrow$

$C(t) = (C_0 - r/k)e^{-kt} + r/k$.

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t\to\infty} C(t) = r/k$.

As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

40. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\frac{10 - x(t)}{10}\,0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\frac{dx}{dt} = \frac{10 - x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\frac{dx}{10 - x} = 0.005\,dt \Rightarrow \frac{-dx}{10 - x} = -0.005\,dt \Rightarrow \ln(10 - x) = -0.005t + c \Rightarrow 10 - x = Ce^{-0.005t}$,

where $C = e^c \Rightarrow x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10(1 - e^{-0.005t})$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$9 = 10(1 - e^{-0.005t}) \Rightarrow 0.9 = 1 - e^{-0.005t} \Rightarrow e^{-0.005t} = 0.1 \Rightarrow -0.005t = -\ln 10 \Rightarrow$

$t = 200 \ln 10 \approx 460.517$ days ≈ 1.26 years.

41. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and $\frac{dy}{dt} = -\left[\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right]\left(10\,\frac{\text{L}}{\text{min}}\right) = -\frac{y(t)}{100}\,\frac{\text{kg}}{\text{min}}$.

$\int \frac{dy}{y} = -\frac{1}{100}\int dt \Rightarrow \ln y = -\frac{t}{100} + C$, and $y(0) = 15 \Rightarrow \ln 15 = C$, so $\ln y = \ln 15 - \frac{t}{100}$.

It follows that $\ln\left(\frac{y}{15}\right) = -\frac{t}{100}$ and $\frac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.