Calculus & Analytic Geometry III

Week 3 HW—Taylor Series

These problems use the techniques of TN §5. Each problem can be derived from the basic series given in Examples 4.2. **Hand in your thoughtful, well-written, and reflective solution to problem 10 on Thursday, April 17.**

Directions
(a) Find the Taylor series for $f(x)$ based at b. Your answer should have one Sigma ($\sum$) sign. On some problems you might want to describe the coefficients using a multi-part notation as in Example 5.5.

(b) Then write the solution in expanded form: $a_0 + a_1(x-b) + a_2(x-b)^2 + \ldots$ where you write at least the first three non-zero terms explicitly.

(c) Then give an interval I where the Taylor series converges.

1. $f(x) = \cos(3x^2)$ based at $b = 0$.

   By substitution. Begin with the Taylor series for $\cos(x)$ at $b = 0$:

$$T(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

   and evaluate for an input of $3x^2$

$$\begin{aligned} \cos(3x^2) = T(3x^2) &= 1 - \frac{(3x^2)^2}{2!} + \frac{(3x^2)^4}{4!} - \frac{3x^2)^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{4n}}{(2n)!}. \end{aligned}$$

   Since the Taylor series for $\cos(x)$ converges on $(-\infty, \infty)$, so will the one for $\cos(3x^2)$.

2. $f(x) = \sin^2(x)$ based at $b = 0$.

   There are several approaches here. (Brute force calculation, multiplying two series for sine, or using a trig identity). We will exploit the double angle formula $\sin^2 x = \frac{1-cos(2x)}{2}$. As above, we find the series for $\cos(2x)$ by substitution.

$$\begin{aligned} \cos(2x) &= 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{2x)^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}. \end{aligned}$$

So

$$
\begin{aligned}
1-\cos(2x) &= \frac{(2x)^2}{2!}-\frac{(2x)^4}{4!}+\frac{2x)^6}{6!}+\cdots \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2x)^{2n}}{(2n)!} \\
\text{and } \frac{1-\cos(2x)}{2} &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2x)^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n-1}x^{2n}}{(2n)!}
\end{aligned}
$$

Since the Taylor series for $\cos(x)$ converges on $(-\infty,\infty)$, so will this one.

3. $f(x)=e^{4x-5}$ based at $b=2$.

   To shift the Taylor series to be about $b=2$, we let $u=x-2$. Then

$$
e^{4x-5}=e^{4(u+2)-5}=e^3e^{4u}=e^3\sum_{k=0}^{\infty}\frac{(4u)^k}{k!}=\sum_{k=0}^{\infty}\frac{e^34^k}{k!}(x-2)^k.
$$

   Since the Taylor series for $\cos(x)$ converges on $(-\infty,\infty)$, so will this one.

4. $f(x)=\sin(x)$ based at $b=\pi/2$.

   Notice that $\cos(x-\pi/2)=\cos x\cos\pi/2+\sin x\sin\pi/2=\sin x$. So

$$
\sin(x)=\cos(x-\pi/2)=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-\frac{(x-\pi/2)^6}{6!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/2)^{2n}}{(2n)!}
$$

   Since the Taylor series for $\cos(x)$ converges on $(-\infty,\infty)$, so will this one.

5. $f(x)=\dfrac{1}{4x-5}-\dfrac{1}{3x-2}$ based at $b=0$.

$$
\begin{aligned}
\frac{1}{4x-5}=\frac{1}{-5(1-\frac{4}{5}x)} &= -\frac{1}{5}\left(1+\left(\frac{4}{5}x\right)+\left(\frac{4}{5}x\right)^2+\cdots+\left(\frac{4}{5}x\right)^n+\cdots\right) \text{ converges for } |x|<5/4 \\
\frac{1}{3x-2}=\frac{1}{-2(1-\frac{3}{2}x)} &= -\frac{1}{2}\left(1+\left(\frac{3}{2}x\right)+\left(\frac{3}{2}x\right)^2+\cdots+\left(\frac{3}{2}x\right)^n+\cdots\right) \text{ converges for } |x|<2/3 \\
\text{So } \frac{1}{4x-5}-\frac{1}{3x-2} &= -\frac{1}{5}\left(1+\left(\frac{4}{5}x\right)+\left(\frac{4}{5}x\right)^2+\cdots+\left(\frac{4}{5}x\right)^n+\cdots\right) \\
&\quad +\frac{1}{2}\left(1+\left(\frac{3}{2}x\right)+\left(\frac{3}{2}x\right)^2+\cdots+\left(\frac{3}{2}x\right)^n+\cdots\right) \\
&= \sum_{k=0}^{\infty}-\frac{1}{5}\left(\frac{4}{5}\right)^k x^k+\frac{1}{2}\left(\frac{3}{2}\right)^k x^k \\
&= \sum_{k=0}^{\infty}\left(-\frac{4^k}{5^{k+1}}+\frac{3^k}{2^{k+1}}\right)x^k.
\end{aligned}
$$

   This converges on the smallest interval, namely $|x|<2/3$.

6. $f(x) = \dfrac{x}{(2x+1)(3x-1)}$ based at $b = 1$.

First translate using $u = x - 1$:

$$f(x) = \frac{x}{(2x+1)(3x-1)} = \frac{u+1}{(2(u+1)+1)(3(u+1)-1)} = \frac{u+1}{(2u+3)(3u+2)}.$$

Now perform a partial fraction decomposition to find that

$$\frac{u+1}{(2u+3)(3u+2)} = \frac{1}{5(3+2u)} + \frac{1}{5(2+3u)} = \frac{1}{15(1+\frac{2}{3}u)} + \frac{1}{10(1+\frac{3}{2}u)}.$$

Proceed as in the previous problem. The series will be

$$\sum_{k=0}^{\infty} \frac{1}{15}\left(\frac{-2}{3}u\right)^k + \frac{1}{10}\left(\frac{-3}{2}u\right)^k = \sum_{k=0}^{\infty}\left[\frac{1}{15}\left(\frac{-2}{3}\right)^k + \frac{1}{10}\left(\frac{-3}{2}\right)^k\right](x-1)^k$$

$= \frac{1}{6} - \frac{7(x-1)}{36} + \frac{55}{216}(x-1)^2 - \frac{463(x-1)^3}{1296} + \frac{4039(x-1)^4}{7776} - \frac{35839(x-1)^5}{46656} + \frac{320503(x-1)^6}{279936} - \frac{2876335(x-1)^7}{1679616} + \frac{25854247(x-1)^8}{10077696} - \frac{232557151(x-1)^9}{60466176} + \frac{2092490071(x-1)^{10}}{362797056} + \cdots.$

The interval of convergence will be $|u| < 2/3$ so $|x-1| < 2/3$ or $1/3 < x < 5/3$.

7. The “sinh” and “cosh” functions are used, for example, in electrical engineering, and are defined by $\sinh(x) = (e^x - e^{-x})/2$, and $\cosh(x) = (e^x + e^{-x})/2$. Do questions (a) and (b) above for the function $h(x) = 2\sinh(3x) - 4\cosh(3x)$ based at $b = 0$.

$\sinh(x) = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots + \dfrac{x^{2n+1}}{(2n+1)!} + \cdots$

$\cosh(x) = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots + \dfrac{x^{2n}}{(2n)!} + \cdots$

So $h(x) = 2\sinh(3x) - 4\cosh(3x) = -4 + 6x - 18x^2 + 9x^3 - \dfrac{27x^4}{2} + \dfrac{81x^5}{20} - \dfrac{81x^6}{20} + \dfrac{243x^7}{280} - \dfrac{729x^8}{1120} + \cdots$.

Converges everywhere.

8. $f(x) = \dfrac{1}{(2x-5)^2(3x-1)}$ based at $b = 0$.

First you will want to use partial fractions to express $f(x)$

$f(x) = \dfrac{1}{(2x-5)^2(3x-1)} = \dfrac{2}{13(-5+2x)^2} - \dfrac{6}{169(-5+2x)} + \dfrac{9}{169(-1+3x)}$.

Express each piece as a geometric series and add. You will get something like this

$f(x) = -\dfrac{1}{25} - \dfrac{19x}{125} - \dfrac{297x^2}{625} - \dfrac{4487x^3}{3125} - \dfrac{13477x^4}{3125} - \dfrac{1010967x^5}{78125} - \dfrac{15164953x^6}{390625} - \cdots$

It converges provided $|3x| < 1$ and $|2x/5| < 1$. So we must have $|x| < 1/3$.

9. $f(x) = \ln(1+x^2)$ based at $b = 0$.

$\ln(1+x^2) = x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{3} - \dfrac{x^8}{4} + \dfrac{x^{10}}{5} - \dfrac{x^{12}}{6} + \dfrac{x^{14}}{7} - \dfrac{x^{16}}{8} + \cdots$

converges when $|x| < 1$.

10. $f(x) = \dfrac{2x}{1+x^2}$ based at $b = 0$.

Do this problem in two ways: (a) Find the series for $1/(1+x^2)$ then multiply it by $2x$ and (b) Differentiate $\ln(1+x^2)$ and use problem (9). Your answer to (b) should agree with your answer to (a).

You should get the same answer as above. Just by a different process. This was your homework to write-up.

11. Find the fifth Taylor polynomial based at $b = 0$ for $f(x) = e^x \sin x$ by multiplication of the series for e^x and $\sin x$ (you do not have to find the general term of the product).

$f(x) = e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + \frac{x^9}{22680} + \frac{x^{10}}{113400} + \cdots$

Since both the Taylor polynomials for e^x and $\sin x$ converge everywhere, so will this product.