Calculus & Analytic Geometry III

§13.1 Vector Functions

Definition. If x, y, and z are given as continuous functions

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

over an interval of t-values, then the set of points $(x, y, z) = (f(t), g(t), h(t))$ defined by these equation is a *parametric curve* (sometimes called a *space curve*). The equations are *parametric equations* for the curve.

Examples.

$$\left.\begin{array}{rcl} x & = & 2\cos t \\ y & = & 3\sin t \\ z & = & t \end{array}\right\} \quad 0 \le t \le \pi \qquad\qquad \left.\begin{array}{rcl} x & = & 5+t \\ y & = & 1+4t \\ z & = & 3-2t \end{array}\right\} \quad 0 \le t \le 1$$

An oval helix

The line in 3-D

$\mathbf{r}(t) = (5+t)\mathbf{i} + (1+4t)\mathbf{j} + (3-2t)\mathbf{k}$

from §12.5

The *vector function* $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is the position vector of the point $P(f(t), g(t), h(t))$on the space curve C. The tip of the moving vector $\mathbf{r}$ traces out the space curve—provided everything is continuous.

Questions.

1. Do vector functions make sense with our original definition of function as a rule that assigns one output for each input?

2. What should the domain of a vector function be?

State the natural domains for:

$\mathbf{r}(t) = \langle t^2, t^3, \sqrt{t} \rangle$

$\mathbf{r}(t) = \langle \sin^{-1}(t), \ln(t), 1 \rangle$

3. What should it mean for a vector function to have a limit at a point a?

Find the limits:

$\lim_{t \to 0} \langle t^2, t^3, \frac{\sin t}{t} \rangle$

$\lim_{t \to 0^+} \langle \sin^{-1}(t), \ln(t), 1 \rangle$

4. What should it mean for a vector function to be continuous at a point a?

5. Can you see that we are leading to asking calculus questions about vector functions?

Calculus & Analytic Geometry III

§13.2 Derivatives and Integrals of Vector Functions

Describe the graphs for $0 \leq t \leq 2\pi$ if $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 5t\mathbf{k}$

What should $\mathbf{r}'(t)$ be?

Does this make sense based on the definition of derivative?

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} =$$

$\mathbf{r}'(t)$ is the *tangent vector* to the curve at the point ....

We occasionally want the *unit tangent vector* $T(t) =$

Example. Find the tangent vector, unit tangent vector, and equation of the line tangent to $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 5t\mathbf{k}$ when $t = \pi/2$.

Differentiation Rules are *mostly* what you would expect—with only a few surprises.

$\boxed{3}$ Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] =$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] =$
3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] =$
4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] =$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] =$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t)$ (chain rule)

Book proves 4. Let's verify 3 and 5 for a particular example. Let $f(t) = \frac{1}{t}$, $\mathbf{u}(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{v}(t) = \langle e^{2t}, e^{-2t}, te^t \rangle$.

Any ideas on how we might prove these things?

Of course, wherever we have derivatives...

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$$

Problems. Find the requested integral of the vector function.

1. Let $\mathbf{r}_1(t) = \dfrac{t}{1+t^2}\mathbf{i} + \dfrac{1}{1+t^2}\mathbf{j} + \dfrac{1}{1-t^2}\mathbf{k}$. Find $\displaystyle\int \mathbf{r}_1(t)dt$

2. Let $\mathbf{r}_2(t) = \sec 2t\mathbf{i} + \tan 3t\mathbf{j} + \ln(1-t)\mathbf{k}$. Find $\displaystyle\int \mathbf{r}_2(t)dt$

3. Let $\mathbf{r}_3(t) = \langle \sin(t), \sin(t)\cos(t), \sin^2(t)\rangle$. Find $\displaystyle\int_0^\pi \mathbf{r}_3(t)dt$.

Can you formulate a FTC II for vector functions?