Calculus & Analytic Geometry III

Sequences and Series

The *nicest* functions we have worked with do date have been polynomials. Last quarter we saw that some (deceptively simple) functions were practically impossible to manipulate analytically. The best we could do was use numerical approximations.

Linear approximations have been with us since TQS 124. Quadratic approximations were introduced when we fit second degree polynomials and developed Simpson's rule for numerical integration. Why stop there? The goal of this first investigation, is to understand when we can use higher degree polynomials to replace difficult functions. Before we jump into Taylor polynomials, we need a short tutorial on sequences, series, and radius of convergence.

Sequences §11.1 (Read this section with a willingness to skip the precise definitions like definition $\boxed{2}$ on page 677)

Definition A *sequence* is a function whose domain is the positive integers. We usually write a_n instead of using the function notation $f(n)$.
Notation: $\{a_1, a_2, a_3, \ldots\}$ $\{a_n\}$ $\{a_n\}_{n=1}^{\infty}$

Examples.

1. $a_n = (-1)^n$
2. $\{1, 2, 6, 24, 120, 720, \ldots\}$
3. $\{1 - (0.2)^n\}$

A sequence $\{a_n\}$ has a *limit* L if the terms of a_n get arbitrarily close to L for n sufficiently large. We write

$$\lim_{n\to\infty} a_n = L.$$

Convergent sequence (limit exists)
Divergent sequence (no limit exists)
Important facts to know:

The sequence $a_n = r^n$ converges to 0 if $|r| < 1$ and diverges if $|r| > 1$. (Example 10)
The sequence $a_n = \frac{1}{n^p}$ converges to 0 if $p > 0$.

Determine the convergence or divergence of the three sequences given in the example above.

$\boxed{3}$ **Theorem** If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is a positive integer, then $\lim_{n\to\infty} a_n = L$.

Do the following sequences converge or diverge?
$a_n = \frac{1-e^{-n}}{1+e^{-n}}$ $b_n = \frac{n}{n+1}$ $c_n = \frac{\ln n}{n}$ $d_n = \frac{n!}{n^n}$

Series §11.2

Definition A *series* is the sum of the terms in an infinite sequence $\{a_n\}_{n=1}^{\infty}$.

$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ $\quad$ $\sum_{n=1}^{\infty} a_n$ $\quad$ $\sum a_n$

Does it make sense to talk about an infinite sum?

Given a series $\{a_n\}$, let s_n denote its nth partial sum

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ converges to S, then the series ...

Examples. Do the following series converge or diverge? If they converge, what is the sum?

$$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n \qquad \sum_{n=1}^{\infty} 1 \qquad \sum_{n=1}^{\infty} \frac{1}{n}$$

Geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$, $a \neq 0$ $\qquad$ Telescoping sum $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$

Do you notice any patterns?

A geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

However the converse is NOT TRUE!!! Just because $\lim_{n\to\infty} a_n = 0$ does not guarantee that the series $\sum_{n=1}^{\infty} a_n$ converges.

The bulk of Chapter 11 is dedicated to techniques used to determine the convergence of series. (Integral test, comparison tests, limit comparison test, ratio test, root test). The most important idea is seen is problems 11.2.47-11.2.51. Namely, introducing a variable x in the series and asking for what values of x the series converges. (These problems resemble polynomials except that they have an infinite number of terms. They are called *power series*.)

11.2.47 $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

11.2.48 $\sum_{n=1}^{\infty} (x-4)^n$

11.2.49 $\sum_{n=0}^{\infty} 4^n x^n$

Taylor Series will be a power series used to represent a function as a limit of polynomials.

We will follow the development of the Taylor Series given in the Taylor Notes (available online).

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Tangent Line Error Bound

From TN §1

Warm-up Find the linear approximations for

- $f(x) = e^x$ at $x = 0$
- $g(x) = \sin x$ at $x = 0$
- $h(x) = \ln x$ at $x = 1$

In general, the linear approximation (also called the first Taylor polynomial) for a nice function $f(x)$ at $x = b$ is

Question. What is the error in our general linear approximation for an input x?

Tangent Line Error Bound. If $|f''(t)| < M$ for all t between x and b then

$$|\text{error}| = \big|f(x) - [f(b) + f'(b)(x-b)]\big| \leq \frac{M}{2}|x-b|^2.$$

Bound the error of our First Taylor polynomials on the given intervals:

$f(x) = e^x$ on $[-1, 1]$ $\quad$ $g(x) = \sin(x)$ on $[-.5, .5]$ $\quad$ $h(x) = \ln(x)$ on $[.98, 1.02]$

Your first witten homework is due Thursday. Class website is `http://depts.washington.edu/uwtmath`. There you will also find suggested homework problems and a link to Taylor Notes. Also submit your preferences for office hours at the online WebQ (see front page of class website).