Calculus & Analytic Geometry III

Beyond Linear Approximations—Looking for a Higher Degree

Warm-up. Find $T_1(x)$ (the first Taylor Polynomial) approximating $f(x) = \tan x$ at $b = \pi/4$.

How accurate is $T_1(x)$ on $[\pi/6, \pi/3]$?

Find and interval J containing $\pi/4$ such that the error is at most 0.001 on J.

Approximations: the naive approach Let's improve our approximation function by fitting a quadratic (rather than linear function) at a specified point $x = b$. We will want the values of the functions, first derivatives, and second derivatives to match at b.

Illustration. A quadratic approximation, $P_2(x)$, to $g(x) = \cos x$ near 0 takes the general form $P_2(x) = C_0 + C_1x + C_2x^2$. Find constants C_0, C_1, and C_2 by requiring $g(0) = P_2(0)$, $g'(0) = P_2'(0)$, and $g''(0) = P_2''(0)$,

$P_2(x) = 1 + 0 \cdot x - \frac{1}{2}x^2$.

Generalized details. Using the same idea, find the coefficients of the quadratic approximation $P_2(x) = C_0 + C_1 x + C_2 x^2$ to a *nice* function $f(x)$ near 0.

Question What about higher-degree polynomials? (Still near 0).

For a *nice* function $f(x)$, the nth Taylor approximation at $x = 0$ is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Examples. Find $T_n(x)$ for the following functions for x near 0.

$f(x) = \dfrac{1}{1-x}$ $\qquad$ $g(x) = e^x$ $\qquad$ $h(x) = \cos x$

$T_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$
$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$T_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^m x^{2m} 2m!$ *where* $m = \lfloor \frac{n}{2} \rfloor$.

Question. What about approximations near some point other than 0?

Set up the Taylor polynomial at $x = b$ at $f(b)$ plus error terms which are zero at $x = b$. Write polynomial as powers of $(x - b)$ rather than powers of x.

$$f(x) \approx T_n(x) = C_0 + C_1(x-b) + C_2(x-b)^2 + \cdots + C_n(x-b)^n.$$

Example. Find the coefficients of $T_3(x)$ (the third Taylor Polynomial) approximating $f(x) = \tan x$ at $b = \pi/4$.

$$T_3(x) = C_0 + C_1(x - \pi/4) + C_2(x - \pi/4)^2 + C_3(x - \pi/4)^3$$

Ans: $T_3(x) = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2 + \frac{8}{3}(x - \pi/4)^3$.

For a *nice* function $f(x)$, the nth Taylor approximation at $x = b$ is

$$T_n(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3 + \frac{f^{(4)}(b)}{4!}(x-b)^4 + \cdots + \frac{f^{(n)}(0)}{n!}(x-b)^n.$$

What's the collateral damage? (Or can we determine the error in the nth Taylor polynomial?)

Start from error in $T_1(x)$:

$$f(x) - \underbrace{(f(b) + f'(b)(x-b))}_{T_1(x)} = \int_b^x f''(t)(x-t)dt \qquad \text{(Use integration by parts again...)}$$

Quadratic Approximation Error Bound. If $|f'''(t)| < M$ for all t between x and b then

$$|\text{error}| = \left| f(x) - \underbrace{\left[f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2\right]}_{T_2(x)} \right| \leq \frac{M}{3!}|x-b|^3.$$

Continuing the process of expressing the exact error as an integral,

$$|f(x) - T_n(x)| = \left| \frac{1}{n!} \int_b^x f^{(n+1)}(x)(x-t)^n dt \right|,$$

breaking it down by parts, regrouping, and approximating leads to the following theorem:

Taylor's Inequality. Suppose I is an interval containing b. If $|f^{(n+1)}(t)| < M$ for all t in I then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}.$$