Calculus & Analytic Geometry III

Taking it to the limit—Taylor Series

Warm-up. Find the indicated Taylor Polynomial at $b = 0$ for the following functions:

$T_5(x)$ for $f(x) = e^{2x}$

$T_8(x)$ for $g(x) = \cos(-x)$

$T_3(x)$ for $h(x) = x \sin x$

$T_3(x)$ for $\ell(x) = \dfrac{1}{9 + x^2}$.

Definition. The *Taylor series* for f based at b is

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} f^{(k)}(b)(x-b)^k = \sum_{k=0}^{\infty} f^{(k)}(b)(x-b)^k$$

provided the limit exists.

Analogy *definite integrals:improper integrals::Taylor polynomials:Taylor series*

The real difference is the issue of convergence for each value of x in the Taylor series. Generally, we try to use Taylor's Theorem to show that the error $|f(x) - T_n(x)|$ goes to zero as $n \to \infty$.

TN Example 4.1 (p. 16-17) shows why e^x converges to its Taylor series for *every* value of x.

Example. For all x, $\sin x$ converges to its Taylor series at $b = 0$.

$f(x)$	Taylor Series for $f(x)$	Interval of Convergence
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	$(-\infty, \infty)$
$\ln(1-x)$	$-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$	$(-1, 1)$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$\frac{1}{1+x^2}$	$1 - x^2 + x^4 - x^6 + \cdots$	$(-1, 1)$
$(1+x)^k$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$(-1, 1)$

Creating New Taylor Series from Old

Goal. Use ingenuity, algebra, and known series to to avoid tedium of Taylor polynomial calculation.

Substitution.

Find Taylor series expansion for e^{-x^2} at $b = 0$.

Find the Taylor series expansion for $\frac{1}{9+x^2}$ at $b = 0$. Where does it converge?

Simple Multiplication. Find Taylor series expansion for $x^2 \cos(x)$ at $b = 0$.

More Interesting Multiplication. Find the Taylor series expansion for $\frac{\ln(1-x)}{1-x}$ at $b = 0$.

A Little Calculus?

Take derivatives: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ on $(-1, 1)$.

Accumulate from 0: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ on $(-1, 1)$.

Accumulate from 0 using the *More Interesting Multiplication* above.

Applications of Taylor Polynomials and Series

Evaluating functions. Calculators and computers generally use appropriate polynomials (though not necessarily Taylor polynomials to generate answers for trigonometric, logarithmic, and exponential functions. In fact the Taylor series for $\arctan x$ can be used to determine $\pi/4$ to many decimal places of accuracy.

Evaluating limits. Try replacing numerator and denominator with Taylor polynomials (plus error terms)

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

Evaluating pesky integrals. When faced with an analytically impossible integral, we can easily compute with its Taylor Series.

$$\int e^{-x^2} dx \qquad \int \frac{\sin x}{x} dx$$