Calculus & Analytic Geometry III

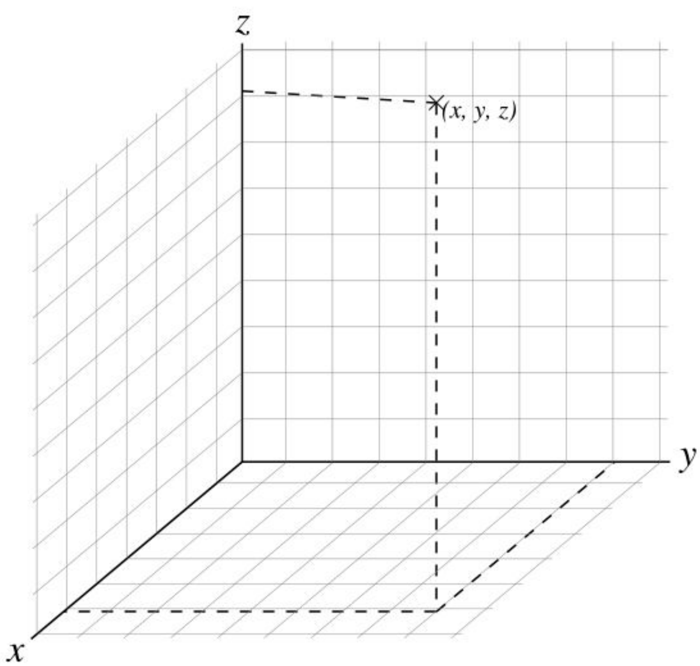
Three-Dimensions: The Preliminaries

Warm-up. Given a $\ell \times w \times h$ rectangular box, what is the length it's diagonal?

Question. How much information do we need to describe a point in 1-dimension? 2-dimensions? 3-dimensions?

coordinate axes
coordinate planes
octants
right-hand rule

$P(x, y, z)$ in three-dimensional rectangular coordinate system (sometimes called Cartesian coordinates).



Distance Formula in Three Dimensions. The distance between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Problems. Find the distance between $P(3, 7, -5)$ and

1. $Q(1, -2, 4)$
2. the x-axis
3. the yz-plane

Find the equation of a sphere centered at P with a radius of 2.

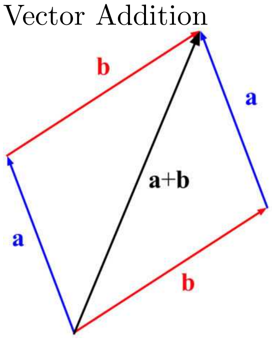
In general, the equation of a sphere with center $C(h,k,l)$ and radius r is

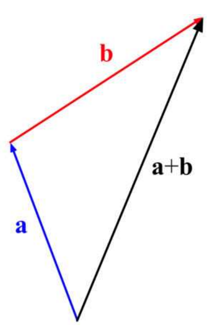
$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2.$$

Find the radius and center of $4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1$.

Definition. A *vector* is a quantity that has a magnitude and a direction. A *scalar* is only a magnitude. (Compare velocity vs. speed)

Vector Addition





Vector Subtraction

Scalar Multiplication

Vector Components. If a particle moves from point $A(a_1, a_2, a_3)$ (the *initial point*) to $B(b_1, b_2, b_3)$ (the *terminal point*), it's displacement vector is

$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.$$

Example. What is the displacement vector $\mathbf{v} = \overrightarrow{AB}$ from $A(3, 1, 4)$ to $B(2, 7, 1)$? What is it's magnitude?

Clarification. We have similar but different notation:

$(1, 2, 3)$ and $\langle 1, 2, 3 \rangle$

Problems. Suppose $\mathbf{a} = \langle 1, 2, -3 \rangle$ and $\mathbf{b} = \langle -2, -1, 5 \rangle$. Find

1. $\mathbf{a} + \mathbf{b}$
2. $2\mathbf{a} - \mathbf{b}$
3. $|\mathbf{a}|$
4. $|\mathbf{a} - \mathbf{b}|$

Properties of Vectors. (page 774) vector addition is commutative and associative, scalar multiplication distributes (sum of scalar distributes across a vector, sum of vectors distributes across a scalar product)

Special Vectors. unit vectors, standard basis vectors

unit vector

$\mathbf{i} =$

$\mathbf{j} =$

$\mathbf{k} =$

So $\mathbf{b} = \langle -2, -1, 5 \rangle =$ ___ $\mathbf{i}+$ ___ $\mathbf{j}+$ ___ $\mathbf{k}$.

Motivating Question. Can we take the product of two vectors? What would it mean?

Dot product (scalar) and cross product (vector).

Definition. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the *dot product* is a number $\mathbf{a} \cdot \mathbf{b}$ given by

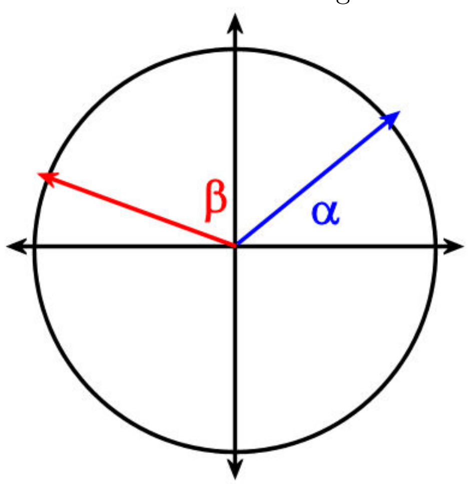
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Find $\mathbf{a} \cdot \mathbf{b}$ if

$\mathbf{a} = \langle 4, 1, 1/4 \rangle$ and $\mathbf{b} = \langle 6, -3, -8 \rangle$

$\mathbf{a} = 4\mathbf{i} + 9\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{a}$ is a unit vector at angle α and $\mathbf{b}$ is a unit vector at angle β.



Given any vector a, what is a unit vector in the same direction?

6 **Corollary** If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

The text uses the law of cosines to prove Theorem 3 and then Corollary 6. You might want to check it out (p. 780).

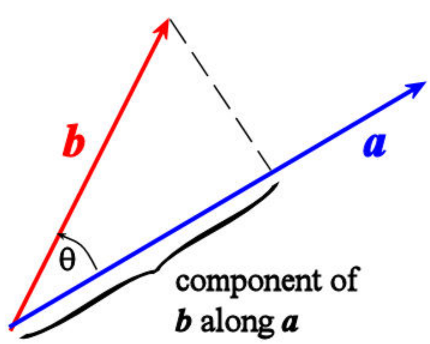
Problem. Find the three angles of the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(2, 2, -3)$.

How do you know when two vectors are perpendicular? How do you know when two vectors are parallel?

Parallel? Perpendicular? or Neither?

$\langle -3, 9, 6\rangle$ and $\langle 4, -12, -8\rangle$ $\quad$ $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ $\quad$ $\langle a, b, c\rangle$ and $\langle -b, a, 0\rangle$

How can we resolve a vector into component parts?



scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{comp}_{\mathbf{a}}\mathbf{b}$

vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{proj}_{\mathbf{a}}\mathbf{b}$

Find the scalar and vector projection of $\mathbf{b}$ onto $\mathbf{a}$ for

$\mathbf{a} = \langle -3, 9, 6\rangle$ and $\mathbf{b} = \langle 4, -12, -8\rangle$ $\quad$ $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

Lots of applications in Mechanics!

A 10 gram block sits perfectly still when placed on a ramp with a 30° incline. What force is friction overcoming to keep the block from moving down the ramp? (Said in another way, what is the projection of the force due to gravity onto a vector in the direction of the ramp?