Calculus & Analytic Geometry III

Vector Products

Scalar:Vector::Dot:Cross

Recall the dot product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

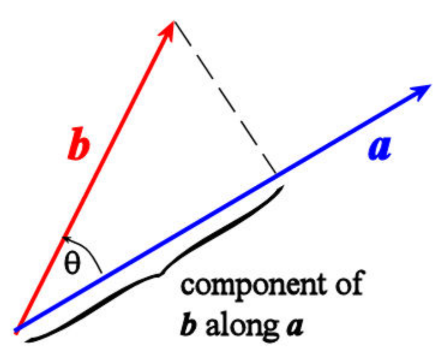
$\mathbf{a} \cdot \mathbf{b} =$

Problem. Find one of the three angles of the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(2, 2, -3)$.

Are the following vectors parallel? perpendicular? or neither?

$\langle -3, 9, 6 \rangle$ and $\langle 4, -12, -8 \rangle$ $\quad$ $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ $\quad$ $\langle a, b, c \rangle$ and $\langle -b, a, 0 \rangle$

How can we resolve a vector into component parts?



scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{comp}_{\mathbf{a}}\mathbf{b}$

vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{proj}_{\mathbf{a}}\mathbf{b}$

Find the scalar and vector projection of $\mathbf{b}$ onto $\mathbf{a}$ for

$\mathbf{a} = \langle -3, 9, 6 \rangle$ and $\mathbf{b} = \langle 4, -12, -8 \rangle$ $\quad$ $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

Lots of applications in Mechanics!

A 10 gram block sits perfectly still when placed on a ramp with a 30° incline. What force is friction overcoming to keep the block from moving down the ramp? (Said in another way, what is the projection of the force due to gravity onto a vector in the direction of the ramp?

The *cross product* of two vectors results in a vector. Only defined for vectors in 3-space.

Geometric Definition If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, the *cross product* $\mathbf{a} \times \mathbf{b}$ is a vector use length equals the area of the parallelogram with edges $\mathbf{a}$ and $\mathbf{b}$ times a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ given by the right hand rule.

Algebraic Definition For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.$$

Investigation. Find $\mathbf{i} \times \mathbf{k}$ geometrically and algebraically. Then find $\mathbf{k} \times \mathbf{i}$.

Cross product is not commutative!!! $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$.

Mnemonic for Cross Product. Use "determinant"

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Problems

1. Find $\langle 1, -1, -1 \rangle \times \langle \frac{1}{2}, 1, \frac{1}{2} \rangle$ and verify that it is orthogonal to both of the given vectors.
2. Find a unit vector orthogonal to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{k}$.
3. Find the area of the triangle PQR if $P(0, -2, 0)$, $Q(4, 1, -2)$, and $R(5, 3, 1)$.
4. Find the volume of the parallelepiped determined by the vectors $\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{i} - \mathbf{j} + \mathbf{k}$, and $-\mathbf{i} + \mathbf{j} + \mathbf{k}$

8 **Theorem** (Properties of cross products)

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Calculus & Analytic Geometry III

Linear Equations

Linear Equations in 2-dimensions. Lines. Constant rate of change.

Algebraic version $y = y_0 + m(x - x_0)$ where m is the slope and (x_0, y_0) is a point on the line.

Vector version (parametric, input t)

$$\langle x(t), y(t) \rangle = \langle x_0, y_0 \rangle + t\mathbf{v}$$

$$(\mathbf{r} = \mathbf{r}_0 + t\mathbf{v})$$

where $\mathbf{v}$ gives the direction of the line and $\mathbf{r}_0 = \langle x_0, y_0 \rangle$ is a vector from the origin to a point $P(x_0, y_0)$ on the line.

Example. Consider the line $y = 4x - 2$.

Lines in 3-dimensions.

Vector version (parametric, input t)

$$\langle x(t), y(t), z(t)\rangle = \langle x_0, y_0, z_0\rangle + t\mathbf{v}$$

$$(\mathbf{r} = \mathbf{r}_0 + t\mathbf{v})$$

where $\mathbf{v}$ gives the direction of the line and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ is a vector from the origin to a point $P(x_0, y_0, z_0)$ on the line.

Example. Find the equation of the line that passes through the points $P(1, 2, -1)$ and $Q(2, 2, 3)$.

Linear Equations in 3-dimensions. Planes. Constant rates of change along any direction in the surface.

Algebraic version $z = z_0 + m_x(x - x_0) + m_y(y - y_0)$ where m_x is the rate of change in the x-direction, m_y is the rate of change in the y-directions, and (x_0, y_0, z_0) is a point on the plane.

Vector version Given a direction vector perpendicular to the plane $\mathbf{n}$ (called the *normal vector* and a point of the plane (x_0, y_0, z_0)

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) =$$

Example 1. Find the equation of the plane perpendicular to $3\mathbf{i} + -1\mathbf{j} + 2\mathbf{k}$ containing the point $P(1, 1, 2)$.

Example 2. Find the equation of the plane containing points $P(1, 2, -1)$, $Q(2, 2, 3)$, and $R(3, 4, -1)$.

Observation. The general form of a plane in 3-space is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad \text{or} \qquad ax + by + cz = d.$$

In either case the normal vector is $\mathbf{n} =$

Now you are ready to answer all kinds of questions about points, planes, distances, and more in 3-D.

1. Sketch the plane $3x + y + 2z = 6$.
2. Where does the line $x = y - 1 = 2z$ intersect the plane $4x - y + 3z = 8$?
3. Find the line of intersection and the angle between the planes $3x-2y+z-1$ and $2x+y-3z = 3$.
4. Find the distance from the origin to the plane $3x + y + 2z = 6$.
5. Find the distance between the planes $2x - y + z = 1$ and $6x - 3y + 3z = 1$