Calculus & Analytic Geometry III

Calculus on parametric equations in 2- and 3-dimensions

Warm-up. Describe the graphs for $0 \leq t \leq 2\pi$:

$\mathbf{r}(t) = \langle \cos 4t, t, \sin 4t \rangle$

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 5t\mathbf{k}$

Calculus on Plane Curves (§10.2) will depend heavily on the chain rule in Leibniz notation.

If y is a function of x
x is a function is t
then y is a function of t *implicitly.*

$$\frac{dy}{dt} =$$

For parametric equations
y is a function of t
x is a function is t
so y is a function of x *implicitly.*

$$\frac{dy}{dx} =$$

Check your knowledge. Find the slope of the tangent line to the (plane) curve:

$y = \sqrt{1 - x^2}$

$$\left.\begin{array}{rl} x = & \cos t \\ y = & \sin t \end{array}\right\} 0 \leq t \leq \pi$$

Examples. Use graphing strategies from §4.3 and §4.5 to determine the general shape of the plane curves:

$$\left.\begin{aligned} x &= t + \ln t \\ y &= t - \ln t \end{aligned}\right\} t > 0 \qquad \left.\begin{aligned} x &= t - \sin t \\ y &= 1 - \cos t \end{aligned}\right\} 0 \le t \le 4\pi$$

Note: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$.

In the last three examples we could have chosen to present some of the derivatives in vector form.

$\mathbf{r}(t) = \langle \cos t, \sin t\rangle$
$\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$

$\mathbf{r}(t) = \langle t + \ln t, t - \ln t\rangle$
$\mathbf{r}'(t) = \langle 1 + \frac{1}{t}, 1 - \frac{1}{t}\rangle$

$\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$
$\mathbf{r}'(t) = \langle 1 - \cos t, \sin t\rangle$

This idea naturally extends to vector functions in three dimensions. If
$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 5t\mathbf{k}$, what should $\mathbf{r}'(t)$ be?

Does this make sense based on the definition of derivative?
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} =$$

$\mathbf{r}'(t)$ is the *tangent vector* to the curve at the point ....
We occasionally want the *unit tangent vector* $T(t) =$

Example. Find the tangent vector, unit tangent vector, and equation of the line tangent to $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 5t\mathbf{k}$ when $t = \pi/2$.

Differentiation Rules are *mostly* what you would expect—with only a few surprises.

$\boxed{3}$ Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] =$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] =$
3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] =$
4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] =$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] =$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t)$ (chain rule)

Book proves 4. Let's verify 3 and 5 for a particular example. Let $f(t) = \frac{1}{t}$, $\mathbf{u}(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{v}(t) = \langle e^{2t}, e^{-2t}, te^t \rangle$.

Any ideas on how we might prove these things?

Of course, wherever we have derivatives...

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$$

Problems. Find the requested integral of the vector function.

1. Let $\mathbf{r}_1(t) = \frac{t}{1+t^2}\mathbf{i} + \frac{1}{1+t^2}\mathbf{j} + \frac{1}{1-t^2}\mathbf{k}$. Find $\int \mathbf{r}_1(t)dt$
2. Let $\mathbf{r}_2(t) = \sec 2t\mathbf{i} + \tan 3t\mathbf{j} + \ln(1-t)\mathbf{k}$. Find $\int \mathbf{r}_2(t)dt$
3. Let $\mathbf{r}_3(t) = \langle \sin(t), \sin(t)\cos(t), \sin^2(t)\rangle$. Find $\int_0^\pi \mathbf{r}_3(t)dt$.

Can you formulate a FTC II for vector functions?