Calculus & Analytic Geometry III

Arc Length and Curvature

Warm-up. Find the requested integral of the vector function.

- Let $\mathbf{r}_1(t) = \dfrac{t}{1+t^2}\mathbf{i} + \dfrac{1}{1+t^2}\mathbf{j} + \dfrac{1}{1-t^2}\mathbf{k}$. Find $\displaystyle\int \mathbf{r}_1(t)dt$
- Let $\mathbf{r}_3(t) = \langle \sin(t), \sin(t)\cos(t), \sin^2(t)\rangle$. Find $\displaystyle\int_0^\pi \mathbf{r}_3(t)dt$.

Can you formulate a FTC II for vector functions?

What have we done in the past? Areas? Arc length? Surface Area? Just need to figure out the translation to parametric equations...and generalize to vector functions.

Area under a curve.

Cartesian Version:
$y = F(x)$ for $a \le x \le b$

$$A = \int_a^b F(x)dx$$

Parametric Version:
$x = f(t)$, $y = g(t)$ for $\alpha \le t \le \beta$

$$A = \int_\alpha^\beta y dx =$$

Example. Find the area of an ellipse $x = a\cos t$ and $y = b\sin t$, $0 \leq t \leq 2\pi$.

Take care with the limits of integration. Figure them out in terms of x first. Then determine associated parameter values.

Arc Length of Plane Curves

Cartesian Version:

$y = F(x)$ for $a \leq x \leq b$

$$L = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Parametric Version:

$x = f(t)$, $y = g(t)$ for $\alpha \leq t \leq \beta$, $\frac{dx}{dt} > 0$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Find the circumference of an ellipse $x = a\cos t$ and $y = b\sin t$, $0 \leq t \leq 2\pi$.

The length of a space curve is defined exactly the same way. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leq t \leq b$ (or equivalently $x = f(t), y = g(t)$, and $z = h(t)$) where f', g', and h' are continuous and the curve is traversed exactly once as t increases from a to b then

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\mathbf{r}'(t)| dt.$$

Example 1. Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \langle \cos 4t, t, \sin 4t \rangle$ on $0 \leq t \leq 2\pi$.

Example 2. Find the length of the curve $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \leq t \leq 1$.

An Interesting Problem. There are many ways to parameterize the same curve and the tangent vector depends on the parameterization. As an illustration, find the tangent vector at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$ for each of the parameterizations below:

$x(t) = \cos(3t)$	$x(t) = \cos(2t)$
$y(t) = \sin(3t)$	$y(t) = \sin(2t)$
$z(t) = 1$	$z(t) = 1$

But they are both exactly the same curve! How can we remedy the situation? Parameterize in terms of arc length rather than time.

Arc length (accumulation) function. If C is a curve given by the vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad a \leq t \leq b$$

where $\mathbf{r}'$ is continuous and C is only traversed cone,

$$s(t) = \int_a^t |\mathbf{r}'(u)| du$$

accumulates the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.

We can reparameterize each curve in terms of arc length (thinking of t as an implicit function of s).

Examples. Parametrize the following curves $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with respect to arc length from the point where $t = 0$ in the direction of increasing t.

$x(t) = \cos(3t)$	$x(t) = \cos(2t)$	$x(t) = e^{2t}\cos(2t)$
$y(t) = \sin(3t)$	$y(t) = \sin(2t)$	$y(t) = 2$
$z(t) = 1$	$z(t) = 1$	$z(t) = e^{2t}\sin 2t$

Curvature measures how quickly a unit tangent vector changes direction at a point. It is defined to be the magnitude of the rate of change of the unit tangent vector with respect to arc length (and we use arc length to make the curvature independent of parameterization.)

Definition. The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

Using the chain rule we can express the curvature as

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \text{ and also } \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Illustration. Find the curvature of $\mathbf{r}(t) = \cos 3t\mathbf{i} + \sin 3t\mathbf{j} + \mathbf{k}$ in three different ways:

Principle unit normal vector
$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

Binormal Vector
$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.