Calculus & Analytic Geometry III

Functions of Several Variables

Definition. Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's *domain*. The set of w-values taken on by f is the function's *range*. The symbol w is the *dependent variable* of f and f is said to be a function of the n *independent variables* x_1 to x_n. (We also call the x_js the function's *input variables* and w the function's *output variable.*)

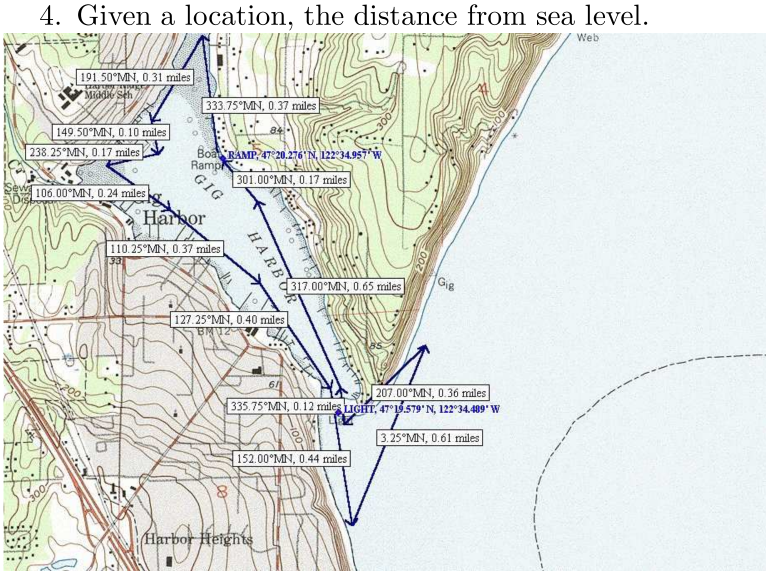
Examples.

1. The sunrise function. Given a longitude, latitude, and date, the sunrise function outputs the time that the sun rises at that location on the given date.
   See `http://www.sunrisesunset.com/`

2. Wind-chill index. Given actual temperature and wind speed, the wind-chill index gives the apparent danger/discomfort of cold weather conditions.
   See `http://www.weatherimages.org/data/windchill.html.`

3. $w = \sqrt{y - x^2}$

4. Given a location, the distance from sea level.



Connection. Compare and contrast functions of two-variables with the vector functions from the previous chapter.

Functions of 2-variables. If f is a function of 2 variables with domain D, then the *graph* of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D. (Called the surface $z = f(x, y)$. The set of all points on the plane where a function has a constant value is called a *level curve* of f.)

Examples. For each of the functions given below, find its domain, range, describe its level curves and its surface.

1. $f(x, y) = y - x$
2. $g(x, y) = \sqrt{25 - x^2 - y^2}$
3. $h(x, y) = e^{-(x^2+y^2)}$
4. $j(x, y) = \sin x + \sin y$

We can see how well we did by visiting `http://www.math.uri.edu/~bkaskosz/flashmo/graph3d/`

Can we generalize to three variables $f(x, y, z)$? What would the equivalent of level curves be?

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Partial Derivatives

Partial derivatives for functions of two variables: the definitions. For a one-variable function, the derivative describes the ratio between output change and input change—*when the input change is small*:

$$f'(a) \approx \frac{\Delta f}{\Delta x} = \frac{f(a+h) - f(a)}{h}, \quad \text{when } \Delta x = h \approx 0$$

The formal **definition** of the derivative is:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For a function of two variables, either one of the input variables can vary. At (a, b), the **partial derivative of a function** $f(x, y)$ **with respect to the variable** x is defined to be the limiting ratio between changes in output and changes in the input x, *with the other input held constant:*

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

(Note that the "prime" notation is not used for partial derivatives. The variable used for differentiation is written as a subscript.)

The **partial derivative of a function** $f(x, y)$ **with respect to** y is defined to be...

Example. For the function $G(x, y) = x^2 + 5\,y^2$, determine the partial derivative $G_x(a, b)$ as follows:

$$G_x(a, b) = \lim_{h \to 0} \frac{((a+h)^2 + 5\,b^2)) - (a^2 + 5\,b^2)}{h}$$

Use this same idea to determine the partial derivative $G_y(a, b)$

In fact, partial derivative with respect to one variable is simply an ordinary derivative where the other variables are treated as constants.

Example. Determine $f_x(x,y)$ and $f_y(x,y)$ for the following functions of two variables:

$$f(x,y) = xy^3 \qquad g(x,y) = \frac{2y}{y+\cos x} \qquad h(x,y) = x\sin xy$$

What does it all mean anyways?

Second Partial Derivatives

$$\begin{aligned}
f_{xx} = (f_x)_x &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\
f_{xy} = (f_x)_y &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \\
f_{yx} = (f_y)_x &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \\
f_{yy} = (f_y)_y &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Clairaut's Theorem. If f is define on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then ...