# Calculus & Analytic Geometry III

## Tangent Planes and Linear Approximations

**Motivating Problem.** For $f(x, y) = x^3 \ln(y)$, find $f_x$, $f_{xy}$, $f_y$, and $f_{yx}$.

Find the equation of the line tangent to the curve defined by the intersection of the surface $f(x, y) = x^3 \ln y$ and the plane $y = e$ at the point $(1, e, 1)$.

Find the equation of the line tangent to the curve defined by the intersection of the same surface $f(x, y) = x^3 \ln y$ and the plane $x = 1$ at the point $(1, e, 1)$.

What do the contour lines (level curves) for $f(x) = x^3 \ln y$ look like? What would the level curves look like if we zoomed in at a particular point—say $(1, e, 1)$?

The *tangent plane* to $f(x, y) = x^3 \ln(y)$ at the point $(1, e, 1)$ is the equation of the plane that contains both tangent lines above.

2 Suppose that $f$ has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

When you are asked to find the local linearization of function of 2-variables, they are simply asking for the equation of the tangent plane.

**Example.** Find the local linearization of $f(x, y) = x^2 + y^2$ at the point $(3, 4)$. Use it to estimate $f(2.9, 4.2)$ and $f(2, 2)$. Compare your answers to the true values.

$f(x, y) \approx 25 + 6(x - 3) + 8(y - 4)$
$f(2.9, 4.2) \approx 26$
$f(2, 2) \approx 3$

The **differential**, $df$ (or $dz$), at a point $(a, b)$ is the linear function of independent variables $dx$ and $dy$ given by the formula

$$df = f_x(a, b)dx + f_y(a, b)dy.$$

The differential at a general point is often written as $df = f_x dx + f_y dy$.

Compute the differentials of the following functions:

$f(x, y) = x^2e^{5y}$ $\qquad z = x\sin(xy)$ $\qquad g(x, y) = x\cos(2x)$

Can you imagine what the differential for a function of three-variables might be? Take a stab at it for $f(x, y, z) = x^y z$.

## Directional Derivatives and the Gradient

Partial derivatives give the rate of change of a function in the directions parallel to the coordinate axes. What if we want to understand the rate of change as we travel in *any* direction?

We can always applied our original definition (looking at how the function changes between two points and seeking the limit as those points get closer and closer.)

2 The *directional derivative* of $f$ at $(x_0, y_0)$ is the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}F(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

**Example** Calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(1, 0)$ in the direction of the vector $\mathbf{i} + \mathbf{j}$.

*Ans:* $\sqrt{2}$

Using the tangent plane as an approximation, we see how to calculate the directional derivative without a limit ($\mathbf{u} = \langle a, b \rangle$ is a unit vector):

$$\begin{aligned} D_{\mathbf{u}}f \approx \frac{df}{h} &= \frac{f_x dx + f_y dy}{h} \\ &= \frac{f_x ah + f_y bh}{h} \\ &= f_x a + f_y b \\ &= \langle f_x, f_y \rangle \cdot \langle a, b \rangle \end{aligned}$$

**Definition.** If $f$ is a function of two variables $x$ and $y$, then the **gradient** of $f$ is the vector function $\nabla f$ defined by

$$\nabla f = \langle f_x, f_y \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

**Problems.** Use the gradient to find the directional derivation of $f(x, y) = x + e^y$ at the point $(1, 1)$ in the direction of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ given below.

$\mathbf{v}_1 = \langle 1, -1 \rangle$
$\mathbf{v}_2 = \mathbf{i} + 2\mathbf{j}$
$\mathbf{v}_3 = \langle 1, 3 \rangle$

**Questions.**

1. In what direction does the largest directional derivative occur?
2. In what direction does no change in the directional derivative occur?
3. If $f$ is the function that gives altitude above sea level for any point on the earth, where on our surface is the gradient $\nabla f = \langle 0, 0 \rangle$?