Calculus & Analytic Geometry III

Application of Partial Derivatives: Finding Extreme Values

Relative Extremes.

1 A function $f(x,y)$ has a *local maximum* at (a,b) if $f(x,y) \leq f(a,b)$ when (x,y) is near (i.e. in an open disk containing) (a,b).

A function $f(x,y)$ has a *local minimum* at (a,b) if $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b).

Critical points are the candidates for local extremes. They occur where ($f_x(a,b) = 0$ or does not exists) AND ($f_y(a,b) = 0$ or does not exist).

Example. Find the critical points for the following functions:

$f(x,y) = 4 + x^3 + y^3 - 3xy$ $\qquad$ $g(x,y) = x^2 + 4y^2 - 2x^2y + 4$

3 **Second Derivatives Test** Suppose that (a,b) is a critical point of $f(x,y)$ and that the second order partial derivatives are continuous in some region that contains (a,b). Dfine,

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

We then have the following classifications of the critical point.

If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a relative minimum.

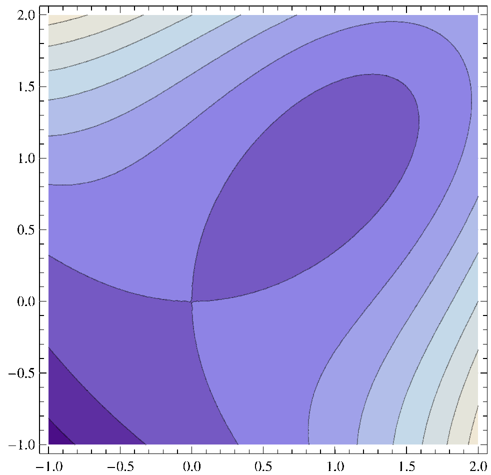
If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a relative maximum.

If $D < 0$, $f(a,b)$ is a saddle point.

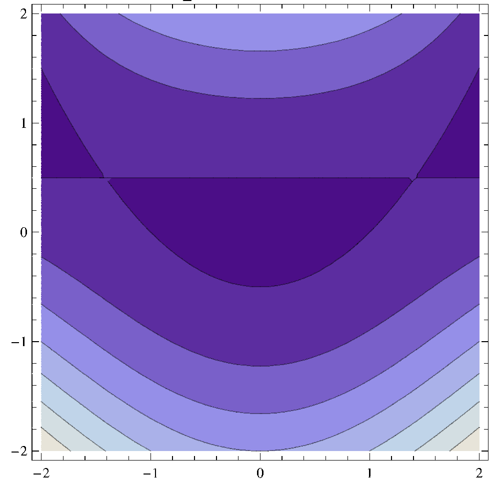
If $D = 0$, then $f(a,b)$ be a relative minimum, relative maximum or a saddle point. Other techniques would need to be used to classify the critical point.

Now classify the critical points found in the previous problems.

$$f(x,y) = 4 + x^3 + y^3 - 3xy$$
$(0,0)$: saddle
$(1,1)$: minimum



$$g(x,y) = x^2 + 4y^2 - 2x^2y + 4$$
$(0,0)$: minimum
$(\sqrt{2}, \frac{1}{2})$: saddle
$(-\sqrt{2}, \frac{1}{2})$: saddle



Absolute Extremes.

Closed and bounded regions in $\mathbb{R}^2$.
closed=contains all boundary points
bounded=contained inside a disk

8 **Extreme Value Theorem for functions of two variables.** If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains absolute maximum and absolute minimum values at some points in D.

Compare methods when function has one vs. two inputs:

Single input variable
Find critical points in interval
Test for max and min at critical points and endpoints

Two input variables
Find critical points in region
Find extreme points on the boundary
Test for max and min at critical points and boundary extremes

Problems.

Find maximum value of

$$f(x,y) = 4 + x^3 + y^3 - 3xy$$

on the disk $x^2 + y^2 \leq 1$.

Find the maximum value of

$$g(x,y) = x^2 + 4y^2 - 2x^2y + 4$$

on the rectangle given by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.