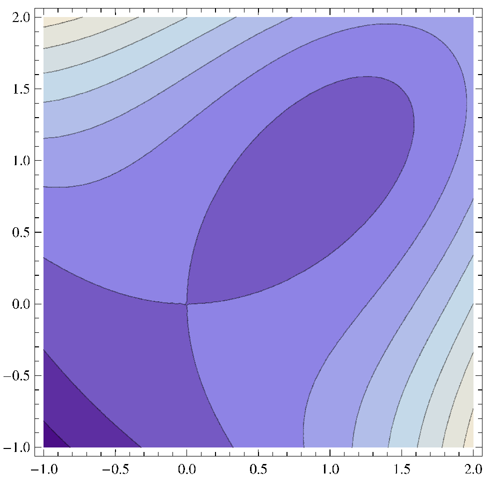
Calculus & Analytic Geometry III

Optimization with constraints: Lagrange Multipliers

From last time. Finding the maximum value of $f(x, y) = 4 + x^3 + y^3 - 3xy$ on the disk $x^2 + y^2 \leq 1$ really boiled down to finding the maximum *on the boundary* $x^2 + y^2 = 1$.

$(0, 0)$: saddle
$(1, 1)$: minimum



We solved for $y = \pm\sqrt{1 - x^2}$ and substituted into the original function to get equations of one variable to maximize:

$$f_1(x) = 4 + x^3 - (1 - x^2)^{3/2} + 3x(1 - x^2)^{1/2}$$

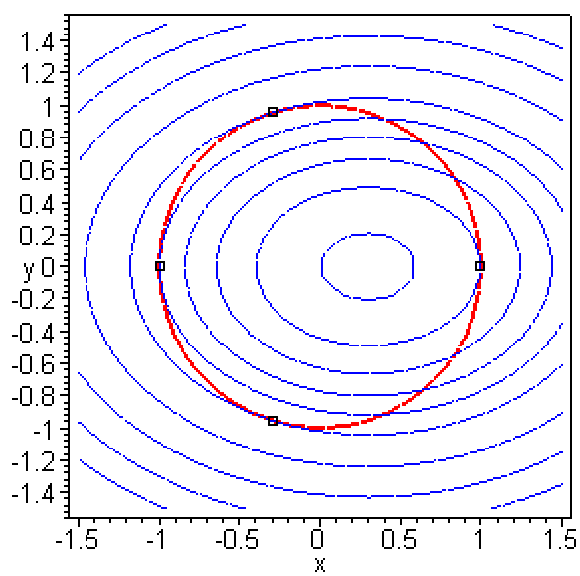
Maximum of 5.83 at $x = -.469$

$$f_2(x) = 4 + x^3 + (1 - x^2)^{3/2} - 3x(1 - x^2)^{1/2}$$

on interval $[-1, 1]$.

Maximum of 5.83 at $x = -.883$

Geometric Approach. Given a function $f(x, y)$ (or even $f(x, y, z)$) subject to a constraint of the form $g(x, y) = k$ (or $g(x, y, z) = k$), how do the level curves of f intersect the constraint curve g? In particular, what can you say about the largest value c such that the level curve $f(x, y) = c$ intersects the constraint curve $g(x, y) = k$?



Extremes can only occur where the level curves of f and the constraint curve *just touch*(i.e. tangent vectors have the same direction!)

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

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Example.

1. Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to $xy = 1$.
2. Find the maximum and minimum values of $f(x, y) = xy$ on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Ans:
1)Minimum of 1 at $(1, 1)$ or $(-1, -1)$. No maximum.
2)Minimum of -2 at $(2, -1)$ and $(-2, 1)$. Maximum of 2 at $(2, 1)$ and $(-2, -1)$

We can use this technique to find extreme values of function subject to two constraints. Suppose we want to find extremes for the function $f(x, y, z)$ subject to constraints $g(x, y, z) = k$ and $h(x, y, z) = c$.

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

Example. The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

Ans:
Closest to origin: $(1, 0, 0)$ and $(0, 1, 0)$.
Farthest from origin: $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2})$.