Calculus & Analytic Geometry III

Definite Integral of a Function of Two Variables

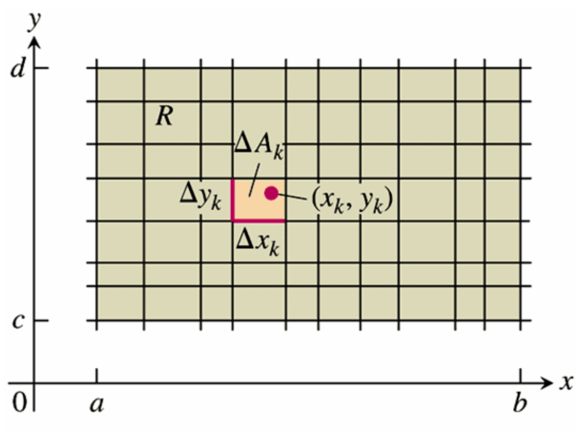
Recall the definition of the definite integral of a function of a single variable:

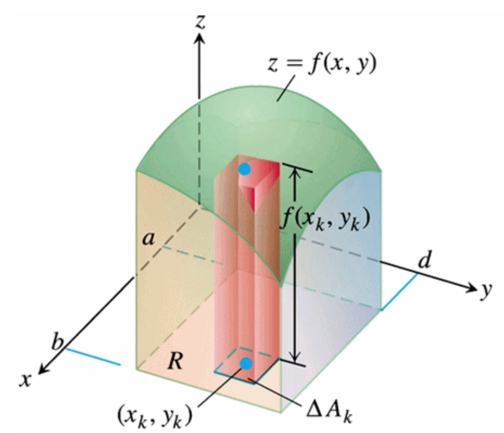
> Let $f(x)$ be defined on $[a,b]$ and let $x_0, x_1, \ldots, x_n$ be a partition of $[a,b]$. For $i = 1, 2, \ldots, n$, let $x_i^* \in [x_{i-1}, x_i]$. Then
>
> $$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Generalizing from one variable to two.

$f(x)$ on an interval $[a,b]$ | $f(x,y)$ on rectangle $R = [a,b] \times [c,d] =$

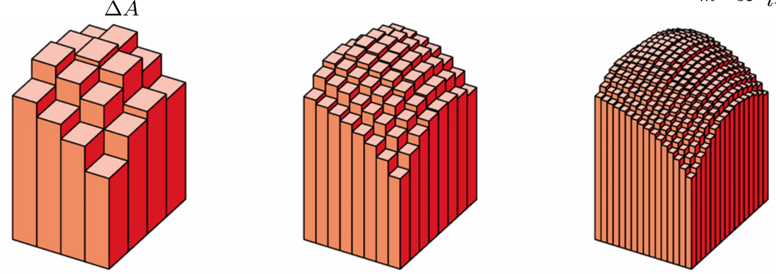
$f(x_i^*)\Delta x$ little bit of area | $f(x_i^*, y_j^*)\Delta x \Delta y$ little bit of volume





Double Riemann Sum If $f(x,y) \geq 0$ the double Riemann sum approximates the volume under the surface.

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*) \underbrace{\Delta x \Delta y}_{\Delta A} \qquad \iint_R f(x,y)dA = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Examples.

1. Let R be the rectangle $1 \le x \le 1.2$ and $2 \le y \le 2.4$. If the values for $f(x, y)$ are as specified below, find Riemann sums which are reasonable over- and under- estimates for $\iint_R f(x,y)dA$ with $\Delta x = 0.1$ and $\Delta y = 0.2$

$y \setminus x$	1.0	1.1	1.2
2.0	5	7	10
2.2	4	6	8
2.4	3	5	6

2. Let $R = [0, 1] \times [0, 1]$. Use Riemann sums to make upper and lower estimates of the volume of the region above R and under the graph $z = e^{-(x^2+y^2)}$.

Other ideas generalize equally as well.

Average Value.

$f(x)$ on an interval $[a, b]$

$$f_{avg}(x) = \frac{1}{b-a}\int_a^b f(x)dx$$

$f(x, y)$ on rectangle $R = [a, b] \times [c, d]$

$$f_{avg}(x,y) = \frac{1}{(b-a)(d-c)}\iint_R f(x,y)dA$$

Obviously we are going to want better methods for evaluating the limits of these double Riemann sums....

Writing a Double Integral as an Iterated Integral If R is a rectangle $a \le x \le b$, $c \le y \le d$, and f is a continuous function on R, then the integral of f over R exists and is equal to the iterated integral

$$\iint_R f(x,y)dA = \int_{y=c}^{y=d}\left(\int_{x=1}^{x=b} f(x,y)dx\right)dy.$$

The expression $\int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)dx\right)dy$ can be written $\int_c^d \int_a^b f(x,y)dx\, dy$.

Work from the inside out...

$$\int_c^d \underbrace{\int_a^b f(x,y)dx}_{\text{Treat } y \text{ as constant}} \quad dy$$

So an iterated integral is a way to transform a double integral into two definite integrals using it partial antiderivatives (that is to say we are reversing the process of partial differentiation.)

Problems. Evaluate the integrals

1. $\int_0^3 \int_0^4 (4x+3y)dx\ dy$
2. $\int_0^4 \int_0^3 (4x+3y)dy\ dx$
3. $\int_1^3 \int_0^4 e^{x+y}dy\ dx$
4. $\int_0^4 \int_1^3 e^{x+y}dx\ dy$

Fubini's Theorem. If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dy\ dx = \int_c^d \int_z^b f(x,y)dx\ dy.$$

Caution. Think before you leap. Sometimes the order of integration can make a huge difference in the ease of calculation. See example 3 page 962 $\iint_R y\sin(xy)dA$ where $R = [1,2] \times [0,\pi]$.

Last Problem if there is time. A building is 8 meters wide and 16 meters long. It has a flat roof that is 12 meters high at one corner and 10 meters high at each of the adjacent corners. What is the volume of the building?

$z = 12 - \frac{1}{4}x - \frac{1}{8}y$.
1280 cubic meters