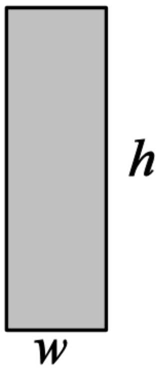
Calculus & Analytic Geometry III

Center of Mass Revisited

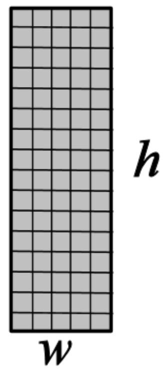
Definition. A *lamina* is thin plate, sheet, or layer.

When a lamina has *uniform density* ρ its center of mass is easier to find.



This is the symmetry principle. If a region is symmetric across a line ℓ then the centroid lies on ℓ.

When lamina has variable density $\rho(x, y)$ we break things up into small rectangles and assume density is constant on each small piece.



$$m = \iint_R \rho(x, y)dA$$
$$M_x = \iint_R y\rho(x, y)dA$$
$$M_y = \iint_R x\rho(x, y)dA$$

Example 1. Suppose a lamina occupies the region D bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ and has density function $\rho(x, y) = y$. Find its mass and center of mass.

$$m = \frac{1}{4}(e^2 - 1)$$
$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{2(e^2 - 1)}, \frac{4(e^3 - 1)}{9(e^2 - 1)}\right)$$

Note: The order of integration is the same for all three calculations m, M_x, and M_y. It can be in any order (dy dx or dx dy), depending on whether the integration is best served by horizontal or vertical slices.

Another Example? Find the center of mass of the thin triangular plate bounded by the x-axis, lines $y = x$, and $y = 2 - x$ if $\rho(x, y) = 6x + 3y + 3$.

Moments of Inertia

A body's first moments (M_x and M_y) tell about balance and torques that the body exerts about the different axes due to gravity. [mass $\times$ distance to axis]

A body's second moments or **moments of inertia** (I_x and I_y) tell about energy stored as the body rotates around different axes. [mass $\times$ square of distance to axis]

$$I_x = \iint_R y^2 \rho(x, y) dA \qquad \text{moment of inertia about } x\text{-axis}$$

$$I_y = \iint_R x^2 \rho(x, y) dA \qquad \text{moment of inertia about } y\text{-axis}$$

$$I_0 = \iint_R (x^2 + y^2) \rho(x, y) dA \qquad \text{moment of inertia about the origin}$$

Example 1 continued. Find the moments of inertia for the lamina bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ with density function $\rho(x, y) = y$.

$I_x = \frac{1}{16}(e^4 - 1)$
$I_y = \frac{1}{8}(e^2 - 1)$
Note: $I_0 = I_x + I_y$

The **radius of gyration of a lamina** with respect to an axis is the second moment's equivalent of the center of mass. Specifically the point $(\overline{\overline{x}}, \overline{\overline{y}})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes.

$$m\overline{\overline{y}}^2 = I_x \qquad m\overline{\overline{x}}^2 = I_y$$

Just as

$$m\overline{y} = M_x \qquad m\overline{x} = M_y.$$

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An abbreviated look at Triple Integrals

1. Evaluate $\displaystyle\int_0^2 \int_0^x \int_0^{x+y} e^x(y+2z)dz\ dy\ dx$.

Helps to know $\displaystyle\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$.

Ans: $19(\frac{e^2}{3}+1)$.

2. Changing the order of integration is sometimes necessary. For example try to evaluate $\displaystyle\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)dz\ dy\ dx$.

Ans: 1.

3. Find the volume of the ellipsoid given by $4x^2+4y^2+z^2=16$.

Ans: $\frac{64\pi}{3}$.