

$$2. \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \mathbf{k}$$

$$= [6 - (-4)]\mathbf{i} - [6 - (-2)]\mathbf{j} + (4 - 2)\mathbf{k} = 10\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$$

Now $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle 10, -8, 2 \rangle \cdot \langle 1, 1, -1 \rangle = 10 - 8 - 2 = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle 10, -8, 2 \rangle \cdot \langle 2, 4, 6 \rangle = 20 - 32 + 12 = 0$,
so $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

$$3. \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \mathbf{k}$$

$$= (15 - 0)\mathbf{i} - (5 - 2)\mathbf{j} + [0 - (-3)]\mathbf{k} = 15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 15 - 9 - 6 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} .

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \cdot (-\mathbf{i} + 5\mathbf{k}) = -15 + 0 + 15 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{b} .

$$7. \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} \mathbf{k}$$

$$= (3t^4 - 2t^4)\mathbf{i} - (3t^3 - t^3)\mathbf{j} + (2t^2 - t^2)\mathbf{k} = t^4\mathbf{i} - 2t^3\mathbf{j} + t^2\mathbf{k}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle t, t^2, t^3 \rangle = t^5 - 2t^5 + t^5 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} .

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle = t^4 - 4t^4 + 3t^4 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{b} .

9. According to the discussion preceding Theorem 8, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, so $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$ [by Example 2].

$$10. \mathbf{k} \times (\mathbf{i} - 2\mathbf{j}) = \mathbf{k} \times \mathbf{i} + \mathbf{k} \times (-2\mathbf{j}) \quad \text{by Property 3 of Theorem 8}$$

$$= \mathbf{k} \times \mathbf{i} + (-2)(\mathbf{k} \times \mathbf{j}) \quad \text{by Property 2 of Theorem 8}$$

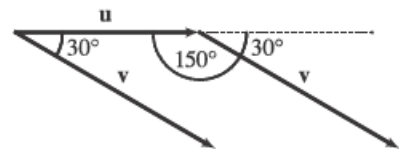
$$= \mathbf{j} + (-2)(-\mathbf{i}) = 2\mathbf{i} + \mathbf{j} \quad \text{by the discussion preceding Theorem 8}$$

15. If we sketch \mathbf{u} and \mathbf{v} starting from the same initial point, we see that

the angle between them is 30° . Using Theorem 6, we have

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin 30^\circ = (6)(8)\left(\frac{1}{2}\right) = 24.$$

By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed into the page.



$$18. \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 0 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{k} = -4\mathbf{i} - 4\mathbf{j} \text{ so}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ -4 & -4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -4 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ -4 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ -4 & -4 \end{vmatrix} \mathbf{k} = 8\mathbf{i} - 8\mathbf{j} - 8\mathbf{k}.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} \mathbf{k} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \text{ so}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 4 \\ 0 & 0 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 0 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 4 \\ 0 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -2 \\ 0 & 0 \end{vmatrix} \mathbf{k} = 8\mathbf{i} - 8\mathbf{j}.$$

Thus $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

19. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$\langle 1, -1, 1 \rangle \times \langle 0, 4, 4 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} \mathbf{k} = -8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}.$$

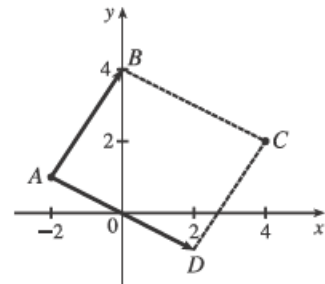
So two unit vectors orthogonal to both are $\pm \frac{\langle -8, -4, 4 \rangle}{\sqrt{64 + 16 + 16}} = \pm \frac{\langle -8, -4, 4 \rangle}{4\sqrt{6}}$, that is, $\left\langle -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$

and $\left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$.

27. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\vec{AB} = \langle 2, 3 \rangle$ and $\vec{AD} = \langle 4, -2 \rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors.

In order to compute the cross product, we consider the vector \vec{AB} as the three-dimensional vector $\langle 2, 3, 0 \rangle$ (and similarly for \vec{AD}), and then the area of parallelogram $ABCD$ is

$$|\vec{AB} \times \vec{AD}| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} \right| = |(0)\mathbf{i} - (0)\mathbf{j} + (-4 - 12)\mathbf{k}| = |-16\mathbf{k}| = 16$$



29. (a) Because the plane through P , Q , and R contains the vectors \overrightarrow{PQ} and \overrightarrow{PR} , a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ} = \langle -1, 2, 0 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(3) - (0)(0), (0)(-1) - (-1)(3), (-1)(0) - (2)(-1) \rangle = \langle 6, 3, 2 \rangle$$

Therefore, $\langle 6, 3, 2 \rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P , Q , and R .

- (b) Note that the area of the triangle determined by P , Q , and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is $|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle 6, 3, 2 \rangle| = \sqrt{36 + 9 + 4} = 7$, so the area of the triangle is $\frac{1}{2}(7) = \frac{7}{2}$.

33. We know that the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is the magnitude of their scalar triple product, which

$$\text{is } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix} = 6(5 + 4) - 3(0 - 8) - (0 - 4) = 82.$$

Thus the volume of the parallelepiped is 82 cubic units.

$$37. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 1 \begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5 \begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2) \begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} = 4 + 60 - 64 = 0, \text{ which says that the volume}$$

of the parallelepiped determined by \mathbf{u} , \mathbf{v} and \mathbf{w} is 0, and thus these three vectors are coplanar.

$$\begin{aligned} 45. (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= (\mathbf{a} - \mathbf{b}) \times \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{b} && \text{by Property 3 of Theorem 8} \\ &= \mathbf{a} \times \mathbf{a} + (-\mathbf{b}) \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + (-\mathbf{b}) \times \mathbf{b} && \text{by Property 4 of Theorem 8} \\ &= (\mathbf{a} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{b}) && \text{by Property 2 of Theorem 8 (with } c = -1) \\ &= \mathbf{0} - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - \mathbf{0} && \text{by Example 2} \\ &= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) && \text{by Property 1 of Theorem 8} \\ &= 2(\mathbf{a} \times \mathbf{b}) \end{aligned}$$

49. (a) No. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$, so \mathbf{a} is perpendicular to $\mathbf{b} - \mathbf{c}$, which can happen if $\mathbf{b} \neq \mathbf{c}$. For example, let $\mathbf{a} = \langle 1, 1, 1 \rangle$, $\mathbf{b} = \langle 1, 0, 0 \rangle$ and $\mathbf{c} = \langle 0, 1, 0 \rangle$.

- (b) No. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ then $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, which implies that \mathbf{a} is parallel to $\mathbf{b} - \mathbf{c}$, which of course can happen if $\mathbf{b} \neq \mathbf{c}$.

- (c) Yes. Since $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$, \mathbf{a} is perpendicular to $\mathbf{b} - \mathbf{c}$, by part (a). From part (b), \mathbf{a} is also parallel to $\mathbf{b} - \mathbf{c}$. Thus since $\mathbf{a} \neq \mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b} - \mathbf{c}$, we have $\mathbf{b} - \mathbf{c} = \mathbf{0}$, so $\mathbf{b} = \mathbf{c}$.