3. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ [since $e^t + e^{-t} > 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

6. $\mathbf{r}(t) = 12t\,\mathbf{i} + 8t^{3/2}\,\mathbf{j} + 3t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 12\,\mathbf{i} + 12\sqrt{t}\,\mathbf{j} + 6t\,\mathbf{k} \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{144 + 144t + 36t^2} = \sqrt{36(t+2)^2} = 6\,|t+2| = 6(t+2)$ for $0 \le t \le 1$. Then

$L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 6(t+2)\,dt = \left[3t^2 + 12t\right]_0^1 = 15.$

8. $\mathbf{r}(t) = \langle t, \ln t, t\ln t\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, 1/t, 1 + \ln t\rangle \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{1^2 + (1/t)^2 + (1 + \ln t)^2} = \sqrt{2 + 1/t^2 + 2\ln t + (\ln t)^2}$ and

$L = \int_1^2 |\mathbf{r}'(t)|\,dt = \int_1^2 \sqrt{2 + 1/t^2 + 2\ln t + (\ln t)^2}\,dt \approx 1.8581.$

13. $\mathbf{r}(t) = 2t\,\mathbf{i} + (1 - 3t)\,\mathbf{j} + (5 + 4t)\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4 + 9 + 16} = \sqrt{29}$. Then $s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$. Therefore, $t = \frac{1}{\sqrt{29}}s$, and substituting for t in the original equation, we have $\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}}s\,\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}s\right)\mathbf{k}$.

15. Here $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$, so $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$. The point $(0,0,3)$ corresponds to $t = 0$, so the arc length function beginning at $(0,0,3)$ and measuring in the positive direction is given by $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t$. $s(t) = 5 \quad\Rightarrow\quad 5t = 5 \quad\Rightarrow\quad t = 1$, thus your location after moving 5 units along the curve is $(3\sin 1, 4, 3\cos 1)$.

19. (a) $\mathbf{r}(t)=\langle\sqrt{2}\,t,e^t,e^{-t}\rangle \Rightarrow \mathbf{r}'(t)=\langle\sqrt{2},e^t,-e^{-t}\rangle \Rightarrow |\mathbf{r}'(t)|=\sqrt{2+e^{2t}+e^{-2t}}=\sqrt{(e^t+e^{-t})^2}=e^t+e^{-t}$.

Then

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{1}{e^t+e^{-t}}\langle\sqrt{2},e^t,-e^{-t}\rangle=\frac{1}{e^{2t}+1}\langle\sqrt{2}e^t,e^{2t},-1\rangle \quad \left[\text{after multiplying by } \frac{e^t}{e^t}\right] \quad \text{and}$$

$$\mathbf{T}'(t)=\frac{1}{e^{2t}+1}\langle\sqrt{2}e^t,2e^{2t},0\rangle-\frac{2e^{2t}}{(e^{2t}+1)^2}\langle\sqrt{2}e^t,e^{2t},-1\rangle$$

$$=\frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\langle\sqrt{2}e^t,2e^{2t},0\rangle-2e^{2t}\langle\sqrt{2}e^t,e^{2t},-1\rangle\right]=\frac{1}{(e^{2t}+1)^2}\langle\sqrt{2}e^t\left(1-e^{2t}\right),2e^{2t},2e^{2t}\rangle$$

Then

$$|\mathbf{T}'(t)|=\frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1-2e^{2t}+e^{4t})+4e^{4t}+4e^{4t}}=\frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1+2e^{2t}+e^{4t})}$$

$$=\frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1+e^{2t}\right)^2}=\frac{\sqrt{2}e^t(1+e^{2t})}{(e^{2t}+1)^2}=\frac{\sqrt{2}\,e^t}{e^{2t}+1}$$

Therefore

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\frac{e^{2t}+1}{\sqrt{2}\,e^t}\,\frac{1}{(e^{2t}+1)^2}\langle\sqrt{2}\,e^t(1-e^{2t}),2e^{2t},2e^{2t}\rangle$$

$$=\frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\langle\sqrt{2}\,e^t(1-e^{2t}),2e^{2t},2e^{2t}\rangle=\frac{1}{e^{2t}+1}\langle 1-e^{2t},\sqrt{2}\,e^t,\sqrt{2}\,e^t\rangle$$

(b) $\kappa(t)=\dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}=\dfrac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\dfrac{1}{e^t+e^{-t}}=\dfrac{\sqrt{2}\,e^t}{e^{3t}+2e^t+e^{-t}}=\dfrac{\sqrt{2}\,e^{2t}}{e^{4t}+2e^{2t}+1}=\dfrac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$

22. $\mathbf{r}(t)=t\,\mathbf{i}+t\,\mathbf{j}+(1+t^2)\,\mathbf{k} \Rightarrow \mathbf{r}'(t)=\mathbf{i}+\mathbf{j}+2t\,\mathbf{k}$, $\mathbf{r}''(t)=2\,\mathbf{k}$, $|\mathbf{r}'(t)|=\sqrt{1^2+1^2+(2t)^2}=\sqrt{4t^2+2}$,

$\mathbf{r}'(t)\times\mathbf{r}''(t)=2\,\mathbf{i}-2\,\mathbf{j}$, $|\mathbf{r}'(t)\times\mathbf{r}''(t)|=\sqrt{2^2+2^2+0^2}=\sqrt{8}=2\sqrt{2}$.

Then $\kappa(t)=\dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}=\dfrac{2\sqrt{2}}{\left(\sqrt{4t^2+2}\right)^3}=\dfrac{2\sqrt{2}}{\left(\sqrt{2}\sqrt{2t^2+1}\right)^3}=\dfrac{1}{(2t^2+1)^{3/2}}$.

30. $y=\ln x \Rightarrow y'=\dfrac{1}{x}$, $y''=-\dfrac{1}{x^2}$,

$\kappa(x)=\dfrac{|y''(x)|}{\left[1+(y'(x))^2\right]^{3/2}}=\left|\dfrac{-1}{x^2}\right|\dfrac{1}{(1+1/x^2)^{3/2}}=\dfrac{1}{x^2}\dfrac{(x^2)^{3/2}}{(x^2+1)^{3/2}}=\dfrac{|x|}{(x^2+1)^{3/2}}=\dfrac{x}{(x^2+1)^{3/2}}$ [since $x>0$].

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$\kappa'(x)=\dfrac{(x^2+1)^{3/2}-x\left(\frac{3}{2}\right)(x^2+1)^{1/2}(2x)}{[(x^2+1)^{3/2}]^2}=\dfrac{(x^2+1)^{1/2}[(x^2+1)-3x^2]}{(x^2+1)^3}=\dfrac{1-2x^2}{(x^2+1)^{5/2}}$;

$\kappa'(x)=0 \Rightarrow 1-2x^2=0$, so the only critical number in the domain is $x=\frac{1}{\sqrt{2}}$. Since $\kappa'(x)>0$ for $0<x<\frac{1}{\sqrt{2}}$

and $\kappa'(x)<0$ for $x>\frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x=\frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}},\ln\frac{1}{\sqrt{2}}\right)$.

Since $\lim_{x\to\infty}\dfrac{x}{(x^2+1)^{3/2}}=0$, $\kappa(x)$ approaches 0 as $x\to\infty$.

31. Since $y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \dfrac{e^x}{(1 + e^{2x})^{3/2}} = e^x(1 + e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$\kappa'(x) = e^x(1 + e^{2x})^{-3/2} + e^x\left(-\frac{3}{2}\right)(1 + e^{2x})^{-5/2}(2e^{2x}) = e^x \dfrac{1 + e^{2x} - 3e^{2x}}{(1 + e^{2x})^{5/2}} = e^x \dfrac{1 - 2e^{2x}}{(1 + e^{2x})^{5/2}}$.

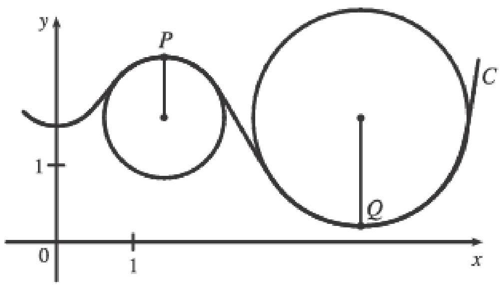
$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and

$1 - 2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

Since $\lim\limits_{x\to\infty} e^x(1 + e^{2x})^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

33. (a) C appears to be changing direction more quickly at P than Q, so we would expect the curvature to be greater at P.

(b) First we sketch approximate osculating circles at P and Q. Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \Rightarrow \kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at Q to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.



43. $\left(1, \frac{2}{3}, 1\right)$ corresponds to $t = 1$. $\quad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\langle 2t, 2t^2, 1\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

$$\mathbf{T}'(t) = -4t(2t^2 + 1)^{-2}\langle 2t, 2t^2, 1\rangle + (2t^2 + 1)^{-1}\langle 2, 4t, 0\rangle \qquad \text{(by Formula 3 of Theorem 14.2 [ET 13.2])}$$
$$= (2t^2 + 1)^{-2}\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\rangle = 2(2t^2 + 1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2 + 1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle}{2(2t^2 + 1)^{-2}\sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} = \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{1 + 2t^2}$$

$\mathbf{N}(1) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left\langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\right\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle$.