4. (a) $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v, which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of h when we fix v and consider h as a function of t, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40,15) = \lim\limits_{h\to 0} \dfrac{f(40+h,15)-f(40,15)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$ and using the values given in the table: $f_v(40,15) \approx \dfrac{f(50,15)-f(40,15)}{10} = \dfrac{36-25}{10} = 1.1$, $f_v(40,15) \approx \dfrac{f(30,15)-f(40,15)}{-10} = \dfrac{16-25}{-10} = 0.9$. Averaging these values, we have $f_v(40,15) \approx 1.0$. Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly, $f_t(40,15) = \lim\limits_{h\to 0} \dfrac{f(40,15+h)-f(40,15)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_t(40,15) \approx \dfrac{f(40,20)-f(40,15)}{5} = \dfrac{28-25}{5} = 0.6$, $f_t(40,15) \approx \dfrac{f(40,10)-f(40,15)}{-5} = \dfrac{21-25}{-5} = 0.8$. Averaging these values, we have $f_t(40,15) \approx 0.7$. Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.

(c) For fixed values of v, the function values $f(v,t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim\limits_{t\to\infty} (\partial h/\partial t) = 0$.

10. $f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x-direction. If we start at $(2,1)$, where $f(2,1) = 10$, and move in the positive x-direction, we reach the next contour line (where $f(x,y) = 12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1) \approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y-direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2,1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1) \approx -2.1$.

15. $f(x,y) = y^5 - 3xy \quad \Rightarrow \quad f_x(x,y) = 0 - 3y = -3y$, $f_y(x,y) = 5y^4 - 3x$

16. $f(x,y) = x^4y^3 + 8x^2y \quad \Rightarrow$

$f_x(x,y) = 4x^3 \cdot y^3 + 8 \cdot 2x \cdot y = 4x^3y^3 + 16xy$, $f_y(x,y) = x^4 \cdot 3y^2 + 8x^2 \cdot 1 = 3x^4y^2 + 8x^2$

21. $f(x,y) = \dfrac{x-y}{x+y} \quad\Rightarrow\quad f_x(x,y) = \dfrac{(1)(x+y)-(x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$,

$f_y(x,y) = \dfrac{(-1)(x+y)-(x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

22. $f(x,y) = x^y \quad\Rightarrow\quad f_x(x,y) = yx^{y-1}$, $f_y(x,y) = x^y \ln x$

40. $f(x,y) = \arctan(y/x) \quad\Rightarrow\quad f_x(x,y) = \dfrac{1}{1+(y/x)^2}\,(-yx^{-2}) = \dfrac{-y}{x^2\,(1+y^2/x^2)} = -\dfrac{y}{x^2+y^2}$,

so $f_x(2,3) = -\dfrac{3}{2^2+3^2} = -\dfrac{3}{13}$.

45. $x^2+y^2+z^2 = 3xyz \quad\Rightarrow\quad \dfrac{\partial}{\partial x}\,(x^2+y^2+z^2) = \dfrac{\partial}{\partial x}\,(3xyz) \quad\Rightarrow\quad 2x + 0 + 2z\,\dfrac{\partial z}{\partial x} = 3y\left(x\,\dfrac{\partial z}{\partial x} + z\cdot 1\right) \quad\Leftrightarrow$

$2z\,\dfrac{\partial z}{\partial x} - 3xy\,\dfrac{\partial z}{\partial x} = 3yz - 2x \quad\Leftrightarrow\quad (2z-3xy)\,\dfrac{\partial z}{\partial x} = 3yz - 2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz-2x}{2z-3xy}$.

$\dfrac{\partial}{\partial y}\,(x^2+y^2+z^2) = \dfrac{\partial}{\partial y}\,(3xyz) \quad\Rightarrow\quad 0 + 2y + 2z\,\dfrac{\partial z}{\partial y} = 3x\left(y\,\dfrac{\partial z}{\partial y} + z\cdot 1\right) \quad\Leftrightarrow\quad 2z\,\dfrac{\partial z}{\partial y} - 3xy\,\dfrac{\partial z}{\partial y} = 3xz - 2y \quad\Leftrightarrow$

$(2z-3xy)\,\dfrac{\partial z}{\partial y} = 3xz - 2y$, so $\dfrac{\partial z}{\partial y} = \dfrac{3xz-2y}{2z-3xy}$.

48. $\sin(xyz) = x + 2y + 3z \quad\Rightarrow\quad \dfrac{\partial}{\partial x}\,(\sin(xyz)) = \dfrac{\partial}{\partial x}\,(x+2y+3z) \quad\Rightarrow\quad \cos(xyz)\cdot y\left(x\,\dfrac{\partial z}{\partial x} + z\right) = 1 + 3\,\dfrac{\partial z}{\partial x} \quad\Leftrightarrow$

$(xy\cos(xyz) - 3)\,\dfrac{\partial z}{\partial x} = 1 - yz\cos(xyz)$, so $\dfrac{\partial z}{\partial x} = \dfrac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3}$.

$\dfrac{\partial}{\partial y}\,(\sin(xyz)) = \dfrac{\partial}{\partial y}\,(x+2y+3z) \quad\Rightarrow\quad \cos(xyz)\cdot x\left(y\,\dfrac{\partial z}{\partial y} + z\right) = 2 + 3\,\dfrac{\partial z}{\partial y} \quad\Leftrightarrow$

$(xy\cos(xyz) - 3)\,\dfrac{\partial z}{\partial y} = 2 - xz\cos(xyz)$, so $\dfrac{\partial z}{\partial y} = \dfrac{2 - xz\cos(xyz)}{xy\cos(xyz) - 3}$.

51. $f(x,y) = x^3y^5 + 2x^4y \quad\Rightarrow\quad f_x(x,y) = 3x^2y^5 + 8x^3y$, $f_y(x,y) = 5x^3y^4 + 2x^4$. Then $f_{xx}(x,y) = 6xy^5 + 24x^2y$,

$f_{xy}(x,y) = 15x^2y^4 + 8x^3$, $f_{yx}(x,y) = 15x^2y^4 + 8x^3$, and $f_{yy}(x,y) = 20x^3y^3$.

56. $v = e^{xe^y} \quad\Rightarrow\quad v_x = e^{xe^y}\cdot e^y = e^{y+xe^y}$, $v_y = e^{xe^y}\cdot xe^y = xe^{y+xe^y}$. Then $v_{xx} = e^{y+xe^y}\cdot e^y = e^{2y+xe^y}$,

$v_{xy} = e^{y+xe^y}(1+xe^y)$, $\quad v_{yx} = xe^{y+xe^y}(e^y) + e^{y+xe^y}(1) = e^{y+xe^y}(1+xe^y)$,

$v_{yy} = xe^{y+xe^y}(1+xe^y) = e^{y+xe^y}(x+x^2e^y)$.

57. $u = x\sin(x+2y) \quad \Rightarrow \quad u_x = x\cdot\cos(x+2y)(1) + \sin(x+2y)\cdot 1 = x\cos(x+2y) + \sin(x+2y)$,

$u_{xy} = x(-\sin(x+2y)(2)) + \cos(x+2y)(2) = 2\cos(x+2y) - 2x\sin(x+2y)$,

$u_y = x\cos\left(x+2y\right)\left(2\right) = 2x\cos(x+2y)$,

$u_{yx} = 2x\cdot(-\sin(x+2y)(1)) + \cos\left(x+2y\right)\cdot 2 = 2\cos(x+2y) - 2x\sin(x+2y)$. Thus $u_{xy} = u_{yx}$.