6. $f(x,y) = x\sin(xy) \quad \Rightarrow \quad f_x(x,y) = x\cos(xy)\cdot y + \sin(xy) = xy\cos(xy) + \sin(xy)$ and $f_y(x,y) = x\cos(xy)\cdot x = x^2\cos(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{3}$, then from Equation 6, $D_{\mathbf{u}}\, f(2,0) = f_x(2,0)\cos\frac{\pi}{3} + f_y(2,0)\sin\frac{\pi}{3} = 0 + 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$.

9. $f(x,y,z) = xe^{2yz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\rangle$

(b) $\nabla f(3,0,2) = \langle 1, 12, 0\rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(3,0,2) = \nabla f(3,0,2)\cdot\mathbf{u} = \langle 1,12,0\rangle\cdot\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}$.

11. $f(x,y) = 1 + 2x\sqrt{y} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 2\sqrt{y}, 2x\cdot\frac{1}{2}y^{-1/2}\right\rangle = \left\langle 2\sqrt{y}, x/\sqrt{y}\right\rangle$, $\nabla f(3,4) = \left\langle 4, \frac{3}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{1}{\sqrt{4^2 + (-3)^2}}\langle 4, -3\rangle = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$, so $D_{\mathbf{u}}\, f(3,4) = \nabla f(3,4)\cdot\mathbf{u} = \left\langle 4, \frac{3}{2}\right\rangle\cdot\left\langle\frac{4}{5}, -\frac{3}{5}\right\rangle = \frac{23}{10}$.

16. $f(x,y,z) = \sqrt{xyz} \quad \Rightarrow$

$$\nabla f(x,y,z) = \left\langle \tfrac{1}{2}(xyz)^{-1/2}\cdot yz, \tfrac{1}{2}(xyz)^{-1/2}\cdot xz, \tfrac{1}{2}(xyz)^{-1/2}\cdot xy\right\rangle = \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}}\right\rangle,$$

$\nabla f(3,2,6) = \left\langle \frac{12}{2\sqrt{36}}, \frac{18}{2\sqrt{36}}, \frac{6}{2\sqrt{36}}\right\rangle = \left\langle 1, \frac{3}{2}, \frac{1}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1+4+4}}\langle -1,-2,2\rangle = \left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right\rangle$, so

$D_{\mathbf{u}}\, f(3,2,6) = \nabla f(3,2,6)\cdot\mathbf{u} = \left\langle 1, \frac{3}{2}, \frac{1}{2}\right\rangle\cdot\left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right\rangle = -\frac{1}{3} - 1 + \frac{1}{3} = -1$.

23. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy)\rangle$, $\nabla f(1,0) = \langle 0,1\rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0,1\rangle$.

28. $f(x,y) = ye^{-xy} \quad \Rightarrow \quad f_x(x,y) = ye^{-xy}(-y) = -y^2e^{-xy}$, $\; f_y(x,y) = ye^{-xy}(-x) + e^{-xy} = (1-xy)e^{-xy}$ and $f_x(0,2) = -4e^0 = -4$, $\; f_y(0,2) = (1-0)e^0 = 1$. If $\mathbf{u}$ is a unit vector which makes an angle θ with the positive x-axis, then $D_{\mathbf{u}} f(0,2) = f_x(0,2)\cos\theta + f_y(0,2)\sin\theta = -4\cos\theta + \sin\theta$. We want $D_{\mathbf{u}} f(0,2) = 1$, so $-4\cos\theta + \sin\theta = 1 \quad\Rightarrow$ $\sin\theta = 1 + 4\cos\theta \quad\Rightarrow\quad \sin^2\theta = (1+4\cos\theta)^2 \quad\Rightarrow\quad 1 - \cos^2\theta = 1 + 8\cos\theta + 16\cos^2\theta \quad\Rightarrow$ $17\cos^2\theta + 8\cos\theta = 0 \quad\Rightarrow\quad \cos\theta(17\cos\theta + 8) = 0 \quad\Rightarrow\quad \cos\theta = 0$ or $\cos\theta = -\frac{8}{17}$. If $\cos\theta = 0$ then $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ but $\frac{3\pi}{2}$ does not satisfy the original equation. If $\cos\theta = -\frac{8}{17}$ then $\theta = \cos^{-1}\left(-\frac{8}{17}\right)$ or $\theta = 2\pi - \cos^{-1}\left(-\frac{8}{17}\right)$ but $\theta = \cos^{-1}\left(-\frac{8}{17}\right)$ is not a solution of the original equation. Thus the directions are $\theta = \frac{\pi}{2}$ or $\theta = 2\pi - \cos^{-1}\left(-\frac{8}{17}\right) \approx 4.22$ rad.

41. Let $F(x,y,z) = x^2 - 2y^2 + z^2 + yz$. Then $x^2 - 2y^2 + z^2 + yz = 2$ is a level surface of F and $\nabla F(x,y,z) = \langle 2x, -4y+z, 2z+y \rangle$.

(a) $\nabla F(2,1,-1) = \langle 4,-5,-1 \rangle$ is a normal vector for the tangent plane at $(2,1,-1)$, so an equation of the tangent plane is $4(x-2) - 5(y-1) - 1(z+1) = 0$ or $4x - 5y - z = 4$.

(b) The normal line has direction $\langle 4,-5,-1 \rangle$, so parametric equations are $x = 2+4t,\ y = 1-5t,\ z = -1-t$, and symmetric equations are $\dfrac{x-2}{4} = \dfrac{y-1}{-5} = \dfrac{z+1}{-1}$.

43. $F(x,y,z) = -z + xe^y \cos z \quad \Rightarrow \quad \nabla F(x,y,z) = \langle e^y \cos z, xe^y \cos z, -1 - xe^y \sin z \rangle$ and $\nabla F(1,0,0) = \langle 1,1,-1 \rangle$.

(a) $1(x-1) + 1(y-0) - 1(z-0) = 0$ or $x + y - z = 1$

(b) $x - 1 = y = -z$