4. In the figure, points at approximately $(-1, 1)$ and $(-1, -1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1, \pm 1)$. Similarly, the point $(1, 0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1, 0)$, $(1, 1)$, and $(1, -1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

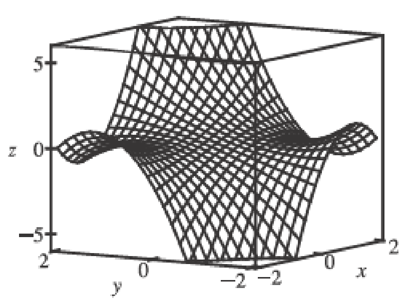
To confirm our predictions, we have $f(x, y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x, y) = 3 - 3x^2$, $f_y(x, y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \Rightarrow x = \pm 1$ and $-4y + 4y^3 = 0 \Rightarrow$ $y(y^2 - 1) = 0 \Rightarrow y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, \pm 1)$. The second partial derivatives are $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = 0$, and $f_{yy}(x, y) = 12y^2 - 4$, so

$D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$.

We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1, 0)$	24	-6	$D > 0$, $f_{xx} < 0 \Rightarrow f$ has a local maximum at $(1, 0)$
$(1, 1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, 1)$
$(1, -1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, -1)$
$(-1, 0)$	-24		$D < 0 \Rightarrow f$ has a saddle point at $(-1, 0)$
$(-1, 1)$	48	6	$D > 0$, $f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, 1)$
$(-1, -1)$	48	6	$D > 0$, $f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, -1)$

9. $f(x, y) = (1 + xy)(x + y) = x + y + x^2y + xy^2 \Rightarrow$ $f_x = 1 + 2xy + y^2$, $f_y = 1 + x^2 + 2xy$, $f_{xx} = 2y$, $f_{xy} = 2x + 2y$, $f_{yy} = 2x$. Then $f_x = 0$ implies $1 + 2xy + y^2 = 0$ and $f_y = 0$ implies $1 + x^2 + 2xy = 0$. Subtracting the second equation from the first gives $y^2 - x^2 = 0 \Rightarrow y = \pm x$, but if $y = x$ then $1 + 2xy + y^2 = 0 \Rightarrow$ $1 + 3x^2 = 0$ which has no real solution. If $y = -x$ then $1 + 2xy + y^2 = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$, so critical points are $(1, -1)$ and $(-1, 1)$.

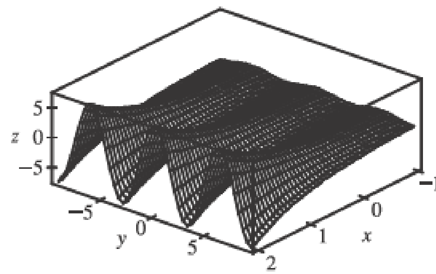


$D(1, -1) = (-2)(2) - 0 < 0$ and $D(-1, 1) = (2)(-2) - 0 < 0$, so $(-1, 1)$ and $(1, -1)$ are saddle points.

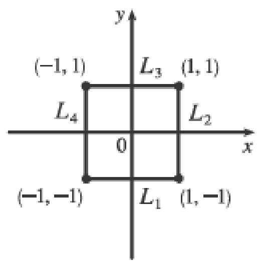
13. $f(x,y) = e^x \cos y \quad \Rightarrow \quad f_x = e^x \cos y,\ f_y = -e^x \sin y.$

Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer.

But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



31. $f_x(x,y) = 2x + 2xy,\ f_y(x,y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0,0)$ as the only critical point in D, with $f(0,0) = 4$.



On L_1: $y = -1$, $f(x,-1) = 5$, a constant.

On L_2: $x = 1$, $f(1,y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1,1)$, $f(1,1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

On L_3: $f(x,1) = 2x^2 + 5$ which attains its maximum at $(-1,1)$ and $(1,1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0,1)$, $f(0,1) = 5$.

On L_4: $f(-1,y) = y^2 + y + 5$ with maximum at $(-1,1)$, $f(-1,1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0,0)$ with $f(0,0) = 4$.

35. $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1-x^2)^2 = x^4 + 2x^3 - 2x^2 + 1, -1 \leq x \leq 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \quad \Rightarrow \quad x = 0$, -2, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2,-3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \leq \theta \leq 2\pi$.

39. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x + y - z = 1$, so $d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$ where $z = x + y - 1$, and we minimize $d^2 = f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2$. Then $f_x(x,y) = 2(x-2) + 2(x+y) = 4x + 2y - 4$, $f_y(x,y) = 2(y-1) + 2(x+y) = 2x + 4y - 2$. Solving $4x + 2y - 4 = 0$ and $2x + 4y - 2 = 0$ simultaneously gives $x = 1$, $y = 0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = 1$, $y = 0$ for which $d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$.