6. $f(x,y)=e^{xy},\ g(x,y)=x^3+y^3=16$, and $\nabla f=\lambda\nabla g \ \Rightarrow\ \langle ye^{xy},xe^{xy}\rangle=\langle 3\lambda x^2,3\lambda y^2\rangle$, so $ye^{xy}=3\lambda x^2$ and $xe^{xy}=3\lambda y^2$. Note that $x=0 \ \Leftrightarrow\ y=0$ which contradicts $x^3+y^3=16$, so we may assume $x\neq 0$, $y\neq 0$, and then $\lambda=ye^{xy}/(3x^2)=xe^{xy}/(3y^2) \ \Rightarrow\ x^3=y^3 \ \Rightarrow\ x=y$. But $x^3+y^3=16$, so $2x^3=16 \ \Rightarrow\ x=2=y$. Here there is no minimum value, since we can choose points satisfying the constraint $x^3+y^3=16$ that make $f(x,y)=e^{xy}$ arbitrarily close to 0 (but never equal to 0). The maximum value is $f(2,2)=e^4$.

7. $f(x,y,z)=2x+6y+10z,\ g(x,y,z)=x^2+y^2+z^2=35 \ \Rightarrow\ \nabla f=\langle 2,6,10\rangle,\ \lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $2\lambda x=2$, $2\lambda y=6$, $2\lambda z=10$ imply $x=\dfrac{1}{\lambda}$, $y=\dfrac{3}{\lambda}$, and $z=\dfrac{5}{\lambda}$. But $35=x^2+y^2+z^2=\left(\dfrac{1}{\lambda}\right)^2+\left(\dfrac{3}{\lambda}\right)^2+\left(\dfrac{5}{\lambda}\right)^2 \ \Rightarrow$ $35=\dfrac{35}{\lambda^2} \ \Rightarrow\ \lambda=\pm 1$, so f has possible extreme values at the points $(1,3,5)$, $(-1,-3,-5)$. The maximum value of f on $x^2+y^2+z^2=35$ is $f(1,3,5)=70$, and the minimum is $f(-1,-3,-5)=-70$.

19. $f(x,y)=e^{-xy}$. For the interior of the region, we find the critical points: $f_x=-ye^{-xy}$, $f_y=-xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0)=1$. For the boundary, we use Lagrange multipliers. $g(x,y)=x^2+4y^2=1 \ \Rightarrow$ $\lambda\nabla g=\langle 2\lambda x,8\lambda y\rangle$, so setting $\nabla f=\lambda\nabla g$ we get $-ye^{-xy}=2\lambda x$ and $-xe^{-xy}=8\lambda y$. The first of these gives $e^{-xy}=-2\lambda x/y$, and then the second gives $-x(-2\lambda x/y)=8\lambda y \ \Rightarrow\ x^2=4y^2$. Solving this last equation with the constraint $x^2+4y^2=1$ gives $x=\pm\frac{1}{\sqrt{2}}$ and $y=\pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{2\sqrt{2}}\right)=e^{1/4}\approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{2\sqrt{2}}\right)=e^{-1/4}\approx 0.779$. The former are the maxima on the region and the latter are the minima.

21. (a) $f(x,y)=x,\ g(x,y)=y^2+x^4-x^3=0 \ \Rightarrow\ \nabla f=\langle 1,0\rangle=\lambda\nabla g=\lambda\langle 4x^3-3x^2,2y\rangle$. Then $1=\lambda(4x^3-3x^2)$ **(1)** and $0=2\lambda y$ **(2)**. We have $\lambda\neq 0$ from **(1)**, so **(2)** gives $y=0$. Then, from the constraint equation, $x^4-x^3=0 \ \Rightarrow\ x^3(x-1)=0 \ \Rightarrow\ x=0$ or $x=1$. But $x=0$ contradicts **(1)**, so the only possible extreme value subject to the constraint is $f(1,0)=1$. (The question remains whether this is indeed the minimum of f.)

(b) The constraint is $y^2+x^4-x^3=0 \ \Leftrightarrow\ y^2=x^3-x^4$. The left side is non-negative, so we must have $x^3-x^4\geq 0$ which is true only for $0\leq x\leq 1$. Therefore the minimum possible value for $f(x,y)=x$ is 0 which occurs for $x=y=0$. However, $\lambda\nabla g(0,0)=\lambda\langle 0-0,0\rangle=\langle 0,0\rangle$ and $\nabla f(0,0)=\langle 1,0\rangle$, so $\nabla f(0,0)\neq\lambda\nabla g(0,0)$ for all values of λ.

(c) Here $\nabla g(0,0)=\mathbf{0}$ but the method of Lagrange multipliers requires that $\nabla g\neq\mathbf{0}$ everywhere on the constraint curve.

27. Let $f(x,y,z)=d^2=(x-2)^2+(y-1)^2+(z+1)^2$, then we want to minimize f subject to the constraint $g(x,y,z)=x+y-z=1$. $\nabla f=\lambda\nabla g \ \Rightarrow\ \langle 2(x-2),2(y-1),2(z+1)\rangle=\lambda\langle 1,1,-1\rangle$, so $x=(\lambda+4)/2$, $y=(\lambda+2)/2$, $z=-(\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2}+\dfrac{\lambda+2}{2}+\dfrac{\lambda+2}{2}=1 \ \Rightarrow$ $3\lambda+8=2 \ \Rightarrow\ \lambda=-2$, so $x=1$, $y=0$, and $z=0$. This must correspond to a minimum, so the shortest distance is $d=\sqrt{(1-2)^2+(0-1)^2+(0+1)^2}=\sqrt{3}$.