Burgers’ Equation Plus Advection

1 Introduction

Consider the nonlinear system

\[ q = \begin{bmatrix} u \\ v \end{bmatrix}, \quad f(q) = \begin{bmatrix} \frac{1}{2}(u^2) \\ (u + 1)v \end{bmatrix}. \quad (1) \]

This is simply Burgers’ equation

\[ u_t + \frac{1}{2}(u^2)_x = 0 \quad (2) \]

coupled to conservative advection

\[ v_t + ((u + 1)v)_x = 0 \quad (3) \]

But note that the advection speed comes from the solution to Burgers’ equation, so there is a 1-way coupling.

2 Analysis as two scalar equations

We can get a feel for what happens by consider the two scalar equations separately. Solving the Burgers’ equation (2) gives a rarefaction wave if \( u_l < u_r \) or a shock wave with speed

\[ s = \frac{1}{2}(u_l + u_r) \]

in the case \( u_l > u_r \).

The advection equation (3) can be rewritten as

\[ v_t + ((u + 1)v)_x = 0 \]

and characteristic theory shows that

\[ \frac{d}{dt} v(X(t), t) = -u_x(X(t), t)v(X(t), t) \]

along the curve \( X'(t) = u(X(t), t) + 1 \). In regions where \( u \) is constant, the characteristics are straight lines, and \( u_x = 0 \) implies that \( v \) is constant.

If the solution \( u \) has a shock wave, then the source term in \( v \) contains a delta function. If the delta moves a different speed than advection velocity, this leads to a jump in \( v \) at the shock location.

If the shock moves at same speed as the advection velocity then the delta function is stationary relative to advecting \( v \) and we expect blow up in the solution for \( v(x, t) \). More generally, if the advection speed \( u_l + 1 \) just to the left of the shock and the speed \( u_r + 1 \) just to the right of the shock satisfy

\[ u_l + 1 > \frac{1}{2}(u_l + u_r) > u_r + 1 \]

then the characteristics of advection would be impinging on the shock from both sides and we’d expect blow up of \( v \). This happens if \( u_r \leq u_l - 2 \).
3 Analysis as a system

Now let’s analyse the system (1) as a system of equations. The Jacobian matrix is

\[ f'(q) = \begin{bmatrix} u & 0 \\ v & u + 1 \end{bmatrix}. \]

The system is always hyperbolic since \( u \neq u + 1 \) and so the matrix can always be diagonalized. The eigenvalues and eigenvectors are

- \( \lambda^1 = u, \ r^1 = \begin{bmatrix} 1 \\ -v \end{bmatrix}, \ \nabla \lambda^1 \cdot r^1 \equiv 1, \ genuinely \ nonlinear \)

and

- \( \lambda^2 = u + 1, \ r^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \nabla \lambda^2 \cdot r^2 \equiv 0, \ linearly \ degenerate \)

3.1 Integral curves and rarefaction waves

The integral curves of \( r^1 \) are curves in the phase plane parameterized by \( \xi \) satisfying

\[ \hat{u}'(\xi) = 0 \ \Rightarrow \ \hat{u}(\xi) = u_* \]
\[ \hat{v}'(\xi) = v(\xi) \ \Rightarrow \ \hat{v}(\xi) = v_* e^{\xi} \]

Since \( u \) is constant, the integral curves are vertical lines in the \( u-v \) phase plane. These lines are also contours of \( \lambda^2 \) (which we expect since this field is linearly degenerate). We’ll see later these are also the Hugoniot loci for 2-waves.

We can solve these ODEs to obtain

\[ \hat{u}'(\xi) = 1 \ \Rightarrow \ \hat{u}(\xi) = u_* + \xi \ \Rightarrow \ \xi = \hat{u} - u_* \]
\[ \hat{v}'(\xi) = -v(\xi) \ \Rightarrow \ \hat{v}(\xi) = v_* e^{-\xi} \ \Rightarrow \ \hat{v} = v_* e^{u_* - \hat{u}}. \]

3.2 Hugoniot loci and shock waves

If \( u_l > u_r \) then we expect a shock in the 1-wave, so we need to find the set of states \( q \) we can connect to some \( q_* \) by a discontinuity satisfying the Rankine-Hugoniot jump condition

\[ f(q) - f(q_*) = s(q - q_*) \]  \( (4) \)

The first equation in this system gives:

\[ \frac{1}{2}(u^2 - u_*^2) = s(u - u_*) \ \Rightarrow \ \frac{1}{2}(u + u_*)(u - u_*) = s(u - u_*). \]  \( (5) \)

One solution is

\( u = u_* \) (and jump in \( v \) arbitrary) \ \Rightarrow \ \text{vertical lines} \)

These are Hugoniot loci for 2-waves. The speed of such a discontinuity can be determined from the second equation in the system (4),

\[ (u + 1)v - (u_* + 1)v_* = s(v - v_*). \]  \( (6) \)
When \( u = u_s \) this reduces to give the speed \( s^2 = u_s + 1 \). These waves are discontinuities in \( v \) alone, propagating with the advection velocity \( u_s + 1 \).

The second possible solution to (5) is

\[
 s = s^1 = \frac{1}{2} (u + u_s).
\]

This is the expected shock speed from Burgers’ equation. The relation between \( v \) and \( u \) across such a shock can be determined from the second equation of R-H relation, (6), with the above value of \( s \):

\[
 (u + 1)v - (u_s - 1)v_s = \frac{1}{2} (u + u_s)(v - v_s)
\]

\[
 \Rightarrow \quad v = \left( \frac{1 + \frac{1}{2}(u_s - u)}{1 - \frac{1}{2}(u_s - u)} \right) v_s \approx e^{u_s - u} v_s
\]

Note that the Hugoniot locus agrees to \( O(|u_s - u|^3) \) with integral curve, as illustrated in Figure 1.

![Image of integral curves and shock Hugoniot](image)

**Figure 1:** Comparison of integral curves of \( r^1(q) \) and shock Hugoniot.

But note that \( v \) blows up along a Hugoniot locus as \( u \to u_s - 2 \):

\[
 v = \left( \frac{1 + \frac{1}{2}(u_s - u)}{1 - \frac{1}{2}(u_s - u)} \right) v_s \to \infty \quad \text{as} \quad u \to u_s - 2
\]

This is not surprising from the discussion of Section 2.

### 4 Solution to the Riemann problem

For general \( q_l = (u_l, v_l) \) and \( q_r = (u_r, v_r) \), the Riemann problem has a classical solution only if \( u_r > u_l - 2 \). Otherwise there is no solution because \( v \) concentrates into a delta function along the shock path \( X(t) = \frac{1}{2}(u_l + u_r)t \).
For cases where there is a Riemann solution, it consists of either a shock followed by a contact discontinuity (if $u_l > u_r > u_l - 2$) or a rarefaction wave followed by a contact discontinuity (if $u_l < u_r$).

4.1 Rarefaction wave solutions

In the case $u_l < u_r$ the Riemann solution consists of a rarefaction wave connecting $q_l$ to $q_m = (u_m, v_m)$ where $u_m = u_r$ and

$$v_m = v_l e^{u_l – u_r}.$$

The left and right edges of the rarefaction fan propagate at velocities $u_l$ and $u_r$ respectively (recall that in $u$ this is a standard rarefaction wave for Burgers’ equation). The rarefaction fan is followed by a faster moving contact discontinuity with speed $s^2 = u_r + 1$ across which $u$ is constant and $v$ jumps from $v_m$ to $v_r$.

Typical cases are shown in Figures 2 and 3. The case shown in 4 contains a transonic rarefaction wave since $u_l < 0 < u_r$.

Figure 2: Riemann solution consisting of a rarefaction wave followed by a contact discontinuity. $q_l = (1, 4)$ and $q_r = (2, -3)$. 
Figure 3: Riemann solution consisting of a rarefaction wave followed by a contact discontinuity. $q_l = (-3, 4)$ and $q_r = (-1.5, 2)$.

Figure 4: Riemann solution consisting of a transonic rarefaction wave followed by a contact discontinuity. $q_l = (-2, 2)$ and $q_r = (2, 3)$. 
4.2 Shock wave solutions

If $u_l > u_r > u_l - 2$ then there is a shock with speed $s^1 = \frac{1}{2}(u_l + u_r)$ connecting $q_l$ to $q_m = (u_m, v_m)$ where $u_m = u_r$ and

\[
v_m = \left( \frac{1 + \frac{1}{2}(u_l - u_r)}{1 - \frac{1}{2}(u_l - u_r)} \right) v_l.
\]

A typical case is shown in Figure 5. Figure 6 shows a case where $u_r$ is closer to $u_l - 2$. Note that $v_m$ is much larger in this case.

5 Clawpack and Python code

The Clawpack code is in subdirectory burgersadv/clawcode.

To best view the codes and plots that result from test runs, move this directory burgersadv to $CLAW/myclaw, start the Clawpack python webserver, and then point your browser to burgersadv/README.html.

The subdirectory burgersadv/python contains a Python module burgersadv.py that is used to create the figures in this document. It can also be used to experiment further with the structure of Riemann solutions.
Figure 6: Riemann solution consisting of a shock followed by a contact discontinuity. $q_l = (3, 2)$ and $q_r = (1.2, 3)$. In this case note that $v_m$ is very large.