One Dimensional QM

Recall \( T = p^2 / 2m \) in classical physics, The de Broglie relation is \( p = \hbar / \lambda = \hbar k. \) \( p = \sqrt{2mE} \) and \( k = 2\pi / \lambda = \sqrt{2mE/\hbar^2} \)

The momentum operator in the \( x \) representation is \( \frac{\hbar}{i} \frac{d}{dx} \)

. The kinetic energy part of the Hamiltonian is \( T = p^2 / 2m = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \)

In general \( H = T + V(x) \) where the potential energy depends on \( x \) but we assume is independent of time.

Notice that the momentum operator, \( p \), does not commute with \( x \); that is \( xp\psi(x) \) is not the same as \( px\psi(x) \) since \( p \) involves a derivative. In fact the commutator is \([x, p] = i\hbar. \) As in the previous case of the spin operators this failure to commute is related to the uncertainty principle relating \( x \) and \( p \).

The Schrodinger equation

\[
H\Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}
\]

becomes

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}
\]

Separate the variables so \( \Psi(x, t) = \psi(x)\phi(t) \).

Then the time equation has the solution

\[
\phi(t) = e^{-iEt/\hbar}
\]

where \( E \), the separation constant, is the eigenvalue of the Hamiltonian, the energy. The \( x \) part of the equation is then

\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi(x) = E \psi(x)
\]

This can be rewritten as

\[
\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2} (E - V(x)) \psi(x) = 0
\]

In the region of space where \( E > V \) the solutions are of the form \( \sin kx \) and \( \cos kx \) while if \( E < V \) the solutions are of the form \( e^{kx} \) and \( e^{-kx} \).

The allowed values of \( k \) and the energy are determined by the boundary conditions. For potentials which do not have an infinite jump the wave function and its derivative are both
continuous. At an infinite jump in potential the derivative is not continuous (for example at the edge of an infinite well).

In the case of a finite depth well the sinusoidal wave function in the well is matched to a decreasing exponential wave function outside the well (the classically forbidden region).

An increasing exponential outside the well is not allowed since then the wave function (and the probability density) would become infinite at large distances.

There is always at least one “bound” state (you can always match a sin function to a decreasing exponential) but the number of allowed states depends on the well depth and width. This matching at the well boundaries imposes a discrete allowed energy spectrum. A finite well has a finite number of allowed energy states. The time dependence of each of the energy states is given by the usual expression. In an energy eigenstate the probability density $\Psi^*\Psi = |\Psi(x,t)|^2$ is independent of time. Why? This is not the case if the system is in a linear combination of energy eigenstates. In that case the probability for finding the particle in different regions varies with time.

If the potential is an even function of $x$ then the wavefunctions will be even or odd functions (for example sin and cos).

**Double Well**

Suppose two identical square wells are separated by an infinite barrier. Then the energy associated with a particle in the ground state in either well is the same. The individual well wavefunctions are $\psi_1(x)$ and $\psi_2(x)$ See Cohen-Tannoudji page 459.

If, however, the potential barrier between the wells is high but finite then a particle will have some probability of being found in either well. So what are the energy eigenstates? They will be approximately the symmetric and antisymmetric (even and odd) combinations $\psi_s(x)$ and $\psi_a(x)$ of the individual well wavefunctions.

If at $t = 0$ a particle is placed in one of the wells the time evolution will be determined by the linear combination of the even and odd energy eigenstates. This situation is exactly analogous to the time evolution of a spin in a magnetic field.

**A Barrier**

Suppose there is a barrier of height $V$ of some width. Everywhere else the potential is zero. Consider an incoming particle with energy $E$, arriving from the left. Now we can specify any incoming energy; this is not a bound system with discrete levels. If $E < V$ there will be exponential wave functions in the classically “forbidden” region within the barrier but sinusoidal wave functions both to the right and left of the barrier.

The wave function and its derivative have to be matched at both edges of the barrier.