3. **Evolution of a spin 1/2 particle in a uniform magnetic field**

From Cohen-Tannoudji

**a. THE INTERACTION HAMILTONIAN AND THE SCHröDINGER EQUATION**

Consider a silver atom in a uniform magnetic field $\mathbf{B}_0$, and choose the Oz axis along $\mathbf{B}_0$. The classical potential energy of the magnetic moment $\mathbf{M} = \gamma \mathbf{S}$ of this atom is then:

$$W = - \mathbf{M} \cdot \mathbf{B}_0 = - M_z B_0 - \gamma B_0 S_z$$  \hspace{1cm} (B-15)

where $B_0$ is the modulus of the magnetic field. Let us set:

$$\omega_0 = - \gamma B_0$$  \hspace{1cm} (B-16)

It is easy to see that $\omega_0$ has the dimensions of the magnetic field. Let us set:

$${\text{It is easy to see that}} \omega_0{\text{ has the dimensions of the inverse of a time, that is, of an angular velocity.}}$$

Since we are quantizing only the internal degrees of freedom of the particle, $\mathcal{H}_0$ must be replaced by the operator $S^+$, and the classical energy (B-15) becomes an operator: it is the Hamiltonian $H$ which describes the evolution of the spin of the atom in the field $\mathbf{B}_0$:

$$H = \omega_0 S_z$$  \hspace{1cm} (B-17)

Since this operator is time-independent, solving the corresponding Schrödinger equation amounts to solving the eigenvalue equation of $H$. We immediately see that the eigenvectors of $H$ are those of $S_z$:

$$H | + \rangle = \frac{\hbar \omega_0}{2} | + \rangle$$  \hspace{1cm} (B-18)

$$H | - \rangle = - \frac{\hbar \omega_0}{2} | - \rangle$$

There are therefore two energy levels, $E_+ = + \hbar \omega_0/2$ and $E_- = - \hbar \omega_0/2$ (fig. 10). Their separation $\hbar \omega_0$ is proportional to the magnetic field; they define a single "Bohr frequency":

$$\nu_{\perp} = \frac{1}{\hbar} (E_+ - E_-) = \frac{\omega_0}{2\pi}$$  \hspace{1cm} (B-19)

**b. Larmor precession**

Let us assume that, at time $t = 0$, the spin is in the state:

$$| \psi(0) \rangle = \cos \frac{\theta}{2} e^{-i \varphi_0 t} | + \rangle + \sin \frac{\theta}{2} e^{i \varphi_0 t} | - \rangle$$  \hspace{1cm} (B-21)

(we showed in § B-1-c that any state could be put in this form). To calculate the state $| \psi(t) \rangle$ at an arbitrary instant $t > 0$, we apply the rule (D-54) given in chapter III. In expression (B-21), $| \psi(0) \rangle$ is already expanded in terms of the eigenvectors of the Hamiltonian, and we therefore obtain:

$$| \psi(t) \rangle = \cos \frac{\theta}{2} e^{-i \varphi_0 t} | + \rangle + \sin \frac{\theta}{2} e^{-i \varphi_0 t} | - \rangle$$  \hspace{1cm} (B-22)

or, using the values of $E_+$ and $E_-$:

$$| \psi(t) \rangle = \cos \frac{\theta}{2} e^{-i \varphi_0 t} | + \rangle + \sin \frac{\theta}{2} e^{i \varphi_0 t} | - \rangle$$

The presence of the magnetic field $\mathbf{B}_0$ therefore introduces a phase shift, proportional to the time, between the coefficients of the kets $| + \rangle$ and $| - \rangle$.

Comparing expression (B-23) for $| \psi(t) \rangle$ with that for the eigenket $| + \rangle$ of the observable $\mathbf{S} \cdot \mathbf{u}$ [formula (A-22-a)], we see that the direction $\mathbf{u}(t)$ along which the spin component is $+ \hbar/2$ with certainty is defined by the polar angles:

$$\begin{cases}
\theta(t) = \theta \\
\varphi(t) = \varphi + \omega_0 t
\end{cases}$$  \hspace{1cm} (B-24)

The angle between $\mathbf{u}(t)$ and Oz (the direction of the magnetic field $\mathbf{B}_0$) therefore remains constant, but $\mathbf{u}(t)$ evolves about Oz at an angular velocity of $\omega_0$ (proportional to the magnetic field). Thus, we find in quantum mechanics the phenomenon which we described for a classical magnetic moment in § A-1-b, and which bears the name of Larmor precession.

From expression (B-17) for the Hamiltonian, it is obvious that the observable $S_z$ is a constant of the motion. It can be verified from (B-23) that the probabilities of obtaining $+ \hbar/2$ or $- \hbar/2$ in a measurement of this observable are time-independent. Since the modulus of $e^{i \theta/2}$ is equal to 1, these probabilities are equal, respectively, to $\cos^2 \theta/2$ and $\sin^2 \theta/2$. The mean value of $S_z$ is also time-independent:

$$\langle \psi(t) | S_z | \psi(t) \rangle = \frac{\hbar}{2} \cos \theta$$  \hspace{1cm} (B-25)
EXERCISES

1. Consider a spin 1/2 particle of magnetic moment \( \mathbf{M} = \gamma \mathbf{S} \). The spin state space is spanned by the basis of the \( |+\rangle \) and \(-\rangle\) vectors, eigenvectors of \( S_z \) with eigenvalues \( +\hbar/2 \) and \(-\hbar/2 \). At time \( t = 0 \), the state of the system is:

\[
|\psi(t=0)\rangle = |+\rangle
\]

a. If the observable \( S_x \) is measured at time \( t = 0 \), what results can be found, and with what probabilities?

b. Instead of performing the preceding measurement, we let the system evolve under the influence of a magnetic field parallel to \( Oy \), of modulus \( B_0 \). Calculate, in the \( \{|+, -\rangle\} \) basis, the state of the system at time \( t \).

c. At this time \( t \), we measure the observables \( S_x, S_y, S_z \). What values can we find, and with what probabilities? What relation must exist between \( B_0 \) and \( t \) for the result of one of the measurements to be certain? Give a physical interpretation of this condition.

2. Consider a spin 1/2 particle, as in the previous exercise (using the same notation).

a. At time \( t = 0 \), we measure \( S_z \) and find \(+\hbar/2\). What is the state vector \( |\psi(0)\rangle \) immediately after the measurement?

b. Immediately after this measurement, we apply a uniform time-dependent field parallel to \( Oz \). The Hamiltonian operator of the spin \( H(t) \) is then written:

\[
H(t) = \omega_0(t) S_z
\]

Assume that \( \omega_0(t) \) is zero for \( t < 0 \) and \( t > T \) and increases linearly from 0 to \( \omega_0 \) when \( 0 \leq t \leq T \) (\( T \) is a given parameter whose dimensions are those of time). Show that at time \( t \) the state vector can be written:

\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left[ e^{i\theta(t)} |+\rangle + i e^{-i\theta(t)} |-\rangle \right]
\]

where \( \theta(t) \) is a real function of \( t \) (to be calculated by the student).

c. At a time \( t = T \), we measure \( S_z \). What results can we find, and with what probabilities? Determine the relation which must exist between \( \omega_0 \) and \( T \) in order for us to be sure of the result. Give the physical interpretation.

3. Consider a spin 1/2 particle placed in a magnetic field \( \mathbf{B}_0 \) with components:

\[
\begin{align*}
B_x &= \frac{1}{\sqrt{2}} B_0 \\
B_y &= 0 \\
B_z &= \frac{1}{\sqrt{2}} B_0
\end{align*}
\]

The notation is the same as that of exercise (1).

a. Calculate the matrix representing, in the \( \{|+, -\rangle\} \) basis, the operator \( H \), the Hamiltonian of the system.

b. Calculate the eigenvalues and the eigenvectors of \( H \).

c. The system at time \( t = 0 \) is in the state \( |-\rangle \). What values can be found if the energy is measured, and with what probabilities?

d. Calculate the state vector \( |\psi(t)\rangle \) at time \( t \). At this instant, \( S_z \) is measured; what is the mean value of the results that can be obtained? Give a geometrical interpretation.

4. Consider the experimental device described in §B.2-b of chapter IV (cf. fig. 8): a beam of atoms of spin 1/2 passes through one apparatus, which serves as a "polarizer" in a direction which makes an angle \( \theta \) with \( Oz \) in the \( xOz \) plane, and then through another apparatus, the "analyzer", which measures the \( S_z \) component of the spin. We assume in this exercise that between the polarizer and the analyzer, over a length \( L \) of the atomic beam, a magnetic field \( \mathbf{B}_0 \) is applied which is uniform and parallel to \( Ox \). We call \( v \) the speed of the atoms and \( T = L/v \) the time during which they are submitted to the field \( \mathbf{B}_0 \). We set \( \omega_0 = -\gamma B_0 \).

a. What is the state vector \( |\psi_1\rangle \) of a spin at the moment it enters the analyzer?

b. Show that when the measurement is performed in the analyzer, there is a probability equal to \( \frac{1}{2} (1 + \cos \theta \cos \omega_0 T) \) of finding \(+\hbar/2\) and \( \frac{1}{2} (1 - \cos \theta \cos \omega_0 T) \) of finding \(-\hbar/2\). Give a physical interpretation.
4.4.2 Electron in a Magnetic Field

A spinning charged particle constitutes a magnetic dipole. Its magnetic dipole moment, $\mu$, is proportional to its spin angular momentum, $S$:

$$\mu = eS;$$  \hspace{1cm}  \text{(4.156)}

the proportionality constant, $e$, is called the gyromagnetic ratio. When a magnetic dipole is placed in a magnetic field $B$, it experiences a torque, $\mu \times B$, which tends to line it up parallel to the field (just like a compass needle). The energy associated with this torque is

$$H = -\mu \cdot B;$$  \hspace{1cm}  \text{(4.157)}

so the Hamiltonian of a spinning charged particle, at rest, in a magnetic field $B$, is

$$H = -\gamma B \cdot S.$$  \hspace{1cm}  \text{(4.158)}

Example 4.3  Larmor precession: Imagine a particle of spin 1/2 at rest in a uniform magnetic field, which points in the $z$-direction:

$$B = B_0 \hat{z}. $$  \hspace{1cm}  \text{(4.159)}

The Hamiltonian (Equation 4.158), in matrix form, is

$$H = -\gamma B_0 S_z = -\frac{\gamma B_0 h}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$  \hspace{1cm}  \text{(4.160)}

The eigenstates of $H$ are the same as those of $S_z$:

$$\begin{align*}
X_+ & = \psi_+ = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \\
X_- & = \psi_- = \left(\begin{array}{c} 0 \\ 1 \end{array}\right). 
\end{align*} $$  \hspace{1cm}  \text{(4.161)}

Evidently the energy is lowest when the dipole moment is parallel to the field—just as it would be classically.

Since the Hamiltonian is time-independent, the general solution to the time-dependent Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi; $$  \hspace{1cm}  \text{(4.162)}

can be expressed in terms of the stationary states:

$$\psi(t) = a X_+ e^{-iE_+ t/\hbar} + b X_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{\frac{i}{\hbar} (\gamma B_0)} \\ b e^{-\frac{i}{\hbar} (\gamma B_0)} \end{pmatrix}. $$

The constants $a$ and $b$ are determined by the initial conditions:

$$\psi(0) = \begin{pmatrix} a \\ b \end{pmatrix}. $$

(of course, $|a|^2 + |b|^2 = 1$. With no essential loss of generality I'll write $a = \cos(\alpha/2)$ and $b = \sin(\alpha/2)$, where $\alpha$ is a fixed angle whose physical significance will appear in a moment. Thus

$$\psi(t) = \begin{pmatrix} \cos(\alpha/2) e^{i B_0 t/\hbar} \\ \sin(\alpha/2) e^{-i B_0 t/\hbar} \end{pmatrix}. $$  \hspace{1cm}  \text{(4.163)}

To get a feel for what is happening here, let's calculate the expectation value of $S_z$ as a function of time:

$$\langle S_z \rangle = \psi(t)^T \hat{S}_z \psi(t) = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)^T \left(\begin{array}{c} \cos(\alpha/2) e^{i B_0 t/\hbar} \\ \sin(\alpha/2) e^{-i B_0 t/\hbar} \end{array}\right)$$

$$\times \langle B_0 / \hbar \rangle = \frac{\hbar}{2} \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \left(\begin{array}{c} \cos(\alpha/2) e^{i B_0 t/\hbar} \\ \sin(\alpha/2) e^{-i B_0 t/\hbar} \end{array}\right)$$

$$= \frac{\hbar}{2} \sin \alpha \cos(\gamma B_0 t). $$

Similarly, 

$$\langle S_y \rangle = \psi(t)^T \hat{S}_y \psi(t) = -\frac{\hbar}{2} \sin \alpha \sin(\gamma B_0 t), $$

and

$$\langle S_x \rangle = \psi(t)^T \hat{S}_x \psi(t) = \frac{\hbar}{2} \cos \alpha. $$

Evidently $(S)$ is tilted at a constant angle $\alpha$ to the $z$-axis, and precesses about the field at the Larmor frequency

$$\omega = \gamma B_0, $$

just as it would classically (see Figure 4.10). No surprise here—Ehrenfest's theorem (in the form derived in Problem 4.20) guarantees that $(S)$ evolves according to the classical laws. But it's nice to see how this works out in a specific context.

![Figure 4.10: Precession of $(S)$ in a uniform magnetic field.](image-url)
In this section we will see an example of how spin can be incorporated into the Schrödinger equation. Imagine that we have a classical magnetic dipole \( \mu \) in a magnetic field \( B \) (Figure 8.7). The magnetic field exerts a torque \( \mu \times B \) on the dipole, which will cause it to line up parallel with the field. But now suppose that in addition, the dipole has angular momentum (Figure 8.8). A rotating body to which a torque is applied will *precess* in a direction perpendicular to the angular momentum vector.

Of course, there is no way to know if these classical analogies will carry over into the quantum realm until we solve the Schrödinger equation. Consider an electron with magnetic moment \( \mu \) at rest in an external magnetic field \( B \). Since we are interested in how the particle spin evolves in time, we will use the time-dependent Schrödinger equation,

\[
H(\psi) = i\hbar \frac{\partial}{\partial t} |\psi\rangle
\]  

(8.26)

\[
H(\psi) = i\hbar \frac{\partial}{\partial t} |\psi\rangle
\]

(8.27)

This equation indicates the form for \( |\psi\rangle \): since \( \sigma_z \) is a 2 \( \times \) 2 Pauli spin matrix, \( |\psi\rangle \) is just the spin state of the electron written as a two-component column vector

\[
|\psi\rangle = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}
\]

where \( \psi_+ \) and \( \psi_- \) will be functions of time. Then Equation (8.27) takes the form

\[
\mu_B B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}
\]

Carrying out the matrix multiplication yields two ordinary differential equations:

\[
\mu_B B \psi_+ = i\hbar \frac{d\psi_+}{dt}
\]

\[
\mu_B B \psi_- = i\hbar \frac{d\psi_-}{dt}
\]

The general solutions of these two equations give the time evolution of \( \psi_+ \) and \( \psi_- \):

\[
\psi_+ = A_+ e^{-i(\mu_B B/\hbar) t}
\]

\[
\psi_- = A_- e^{i(\mu_B B/\hbar) t}
\]

where \( A_+ \) and \( A_- \) are constants to be determined. In matrix form, the solution is then

\[
|\psi\rangle = \begin{pmatrix} A_+ e^{-i(\mu_B B/\hbar) t} \\ A_- e^{i(\mu_B B/\hbar) t} \end{pmatrix}
\]  

(8.28)
The constants $A_+$ and $A_-$ are determined from the initial conditions. In particular, if we take $t = 0$ to be the initial time, then

$$\ket{\psi(t = 0)} = \begin{pmatrix} A_+ \\ A_- \end{pmatrix}$$

For example, suppose that the electron has spin up (i.e., in the $+z$ direction) at $t = 0$. This corresponds to the state

$$\ket{\psi(t = 0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so that $A_+ = 1$ and $A_- = 0$. Then Equation (8.28) gives the wave function at any later time $t$:

$$\ket{\psi} = \begin{pmatrix} e^{-i(\mu_B B/c) t} \\ 0 \end{pmatrix}$$

Although this wave function is a function of time, it represents a state of constant spin. To see this, note that $P = \left| \bra{\uparrow} \ket{\psi(t)} \right|^2$ gives the probability of finding the particle in the spin up state. This probability is

$$P = \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left( e^{-i(\mu_B B/c) t} \\ 0 \end{pmatrix} \right) \right|^2 = 1$$

Thus, the electron starts out in the $\ket{\uparrow}$ state, and it stays there forever. This is consistent with the classical analog: a classical dipole pointing parallel to a magnetic field experiences no torque and does not rotate.

Now consider the more interesting case of an electron with spin initially in the $+x$ direction. In this case the initial spin state (correctly normalized) is

$$\ket{\psi(t = 0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then $A_+ = 1/\sqrt{2}$ and $A_- = 1/\sqrt{2}$, so the full time-dependent wave function from Equation (8.28) is

$$\ket{\psi} = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i(\mu_B B/c) t} \\ \frac{1}{\sqrt{2}} e^{i(\mu_B B/c) t} \end{pmatrix}$$

To simplify the equation, define a new quantity $\omega$ given by

$$\omega = 2\mu_B B/c$$

where $\omega$ has units of $1$/time or, equivalently, frequency. Then the wave function is

$$\ket{\psi} = \frac{1}{\sqrt{2}} \left( e^{-i\omega t/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

To understand the physical meaning of this wave function, we can evaluate it at a variety of times. In particular, we have

$$\ket{\psi(t = 0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which is just our initial condition: the spin is in the $+x$ direction at $t = 0$. Further,

$$\ket{\psi(t = 2\pi/\omega)} = \frac{1}{\sqrt{2}} \left( e^{-i\pi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is the original wave function multiplied by a phase factor of $-1$. Thus, at $t = 2\pi/\omega$, the spin is once again pointing in the $+x$ direction. This suggests that we investigate intermediate times. At the halfway point between $t = 0$ and $t = 2\pi/\omega$, i.e., at $t = (1/2)2\pi/\omega$, we get

$$\ket{\psi(t = (1/2)2\pi/\omega)} = \frac{1}{\sqrt{2}} \left( e^{-i\pi/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ -1 \end{pmatrix}$$

which is the wave function for spin in the $-x$ direction. Taking half of this interval again, we get, at $t = (1/4)(2\pi/\omega)$,

$$\ket{\psi(t = (1/4)2\pi/\omega)} = \frac{1}{\sqrt{2}} \left( e^{-i\pi/4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \frac{1}{2} (1 - i) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which is the spin eigenstate in the $+y$ direction. Similarly, taking $t = (3/4)(2\pi/\omega)$ gives the spin eigenstate in the $-y$ direction.

Putting all of this information together, we see that the electron is precessing, with the direction of its spin vector rotating in the counterclockwise direction (Figure 8.9). The period of precession is $2\pi/\omega$, so the angular frequency is $\omega = 2\mu_B B/c$. This phenomenon is the basis of magnetic resonance imaging, which will be examined in more detail in Chapter 14.

![Figure 8.9](image-url)