Classical Angular Momentum and Torque

A rotating vector

$\vec{A}$ is a vector with fixed magnitude in three dimensional space. It makes an angle $\theta$ with the $z$ axis. $\vec{A}$ rotates around the $z$ axis through an angle $\Delta \phi$. The angle $\phi$ is in the $x, y$ plane.

$$\Delta A = A \sin \theta \Delta \phi$$

$$\frac{\Delta A}{\Delta t} = A \sin \theta \frac{\Delta \phi}{\Delta t}$$

$$\frac{dA}{dt} = A \sin \theta \frac{d\phi}{dt}$$

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

since the angular velocity $\omega = \frac{d\phi}{dt}$. $\vec{\omega}$ is along the axis of rotation.

The direction of $\frac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$ and perpendicular to the axis of rotation.

Torque and force

The torque $\vec{\tau} = \vec{r} \times \vec{F}$

Angular momentum $\vec{L} = \vec{r} \times \vec{p}$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

where we have used $\frac{d\vec{r}}{dt} \times \vec{p} = 0$ since velocity is in same direction as momentum, $p$. 
**Magnetic Moments**

The torque on a magnetic moment, \( \vec{M} \), in a magnetic field, \( \vec{B} \), is

\[
\vec{\tau} = \vec{M} \times \vec{B}
\]

The magnetic moment is related to the spin angular momentum \( \vec{M} = \gamma \vec{S} \), where \( \gamma \) is the gyromagnetic ratio.

Since \( \frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A} \) and \( \vec{\tau} = \frac{d\vec{L}}{dt} \),

\[
\vec{\tau} = \frac{d\vec{S}}{dt} = \frac{1}{\gamma} \frac{d\vec{M}}{dt} = \vec{M} \times \vec{B} = -\vec{B} \times \vec{M} = \gamma \vec{S} \times \vec{B}
\]

\[
\frac{d\vec{M}}{dt} = -\gamma \vec{B} \times \vec{M} \quad \text{or} \quad \frac{d\vec{S}}{dt} = -\vec{B} \times \vec{M}
\]

Use \( \frac{d\vec{M}}{dt} = \vec{\omega} \times \vec{M} \) then the precession angular frequency is

\[
\vec{\omega} = -\gamma \vec{B}
\]