**Warm-up.** Describe the graphs for $0 \leq t \leq 2\pi$:

$$r(t) = \langle \cos 4t, t, \sin 4t \rangle$$

$$r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sin 5t \mathbf{k}$$

**Calculus on Plane Curves** (§10.2) will depend heavily on the chain rule in Leibnitz notation.

If $y$ is a function of $x$

$x$ is a function of $t$

then $y$ is a function of $t$ *implicitly.*

$$\frac{dy}{dt} =$$

For parametric equations

$y$ is a function of $t$

$x$ is a function of $t$

so $y$ is a function of $x$ *implicitly.*

$$\frac{dy}{dx} =$$

**Check your knowledge.** Find the slope of the tangent line to the (plane) curve:

$$y = \sqrt{1 - x^2}$$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} 0 \leq t \leq \pi$$
Examples. Use graphing strategies from §4.3 and §4.5 to determine the general shape of the plane curves:

\[
\begin{align*}
  x &= t + \ln t \quad (t > 0) \\
  y &= t - \ln t
\end{align*}
\]

\[
\begin{align*}
  x &= t - \sin t \\
  y &= 1 - \cos t \quad (0 \leq t \leq 4\pi)
\end{align*}
\]

Note: \( \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right). \)

In the last three examples we could have chosen to present some of the derivatives in vector form.

\[
\begin{align*}
  r(t) &= \langle \cos t, \sin t \rangle \\
  r'(t) &= \langle -\sin t, \cos t \rangle \\
  r(t) &= \langle t + \ln t, t - \ln t \rangle \\
  r'(t) &= \langle 1 + \frac{1}{t}, 1 - \frac{1}{t} \rangle \\
  r(t) &= \langle t - \sin t, 1 - \cos t \rangle \\
  r'(t) &= \langle 1 - \cos t, \sin t \rangle
\end{align*}
\]

This idea naturally extends to vector functions in three dimensions. If

\[
r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sin 5t \mathbf{k},
\]

what should \( r'(t) \) be?

Does this make sense based on the definition of derivative?

\[
\frac{d}{dt} = \frac{r'(t)}{r(t)} = \lim_{h \to 0} \frac{r(t + h) - r(t)}{h} =
\]

\( r'(t) \) is the tangent vector to the curve at the point ....

We occasionally want the unit tangent vector \( T(t) = \)
**Example.** Find the tangent vector, unit tangent vector, and equation of the line tangent to \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sin 5t \mathbf{k} \) when \( t = \pi/2 \).

Differentiation Rules are *mostly* what you would expect—with only a few surprises.

<table>
<thead>
<tr>
<th>3</th>
<th>Suppose ( \mathbf{u} ) and ( \mathbf{v} ) are differentiable vector functions, ( c ) is a scalar, and ( f ) is a real-valued function. Then</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = )</td>
</tr>
<tr>
<td>2.</td>
<td>( \frac{d}{dt}[c\mathbf{u}(t)] = )</td>
</tr>
<tr>
<td>3.</td>
<td>( \frac{d}{dt}[f(t)\mathbf{u}(t)] = )</td>
</tr>
<tr>
<td>4.</td>
<td>( \frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = )</td>
</tr>
<tr>
<td>5.</td>
<td>( \frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = )</td>
</tr>
<tr>
<td>6.</td>
<td>( \frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t) ) (chain rule)</td>
</tr>
</tbody>
</table>

Book proves 4. Let’s verify 3 and 5 for a particular example. Let \( f(t) = \frac{1}{t}, \mathbf{u}(t) = (t, t^2, t^3) \) and \( \mathbf{v}(t) = (e^{2t}, e^{-2t}, te^t) \).

Any ideas on how we might prove these things?
Of course, wherever we have derivatives...

\[ \int_a^b \mathbf{r}(t) dt = \left( \int_a^b f(t) dt \right) \mathbf{i} + \left( \int_a^b g(t) dt \right) \mathbf{j} + \left( \int_a^b h(t) dt \right) \mathbf{k} \]

**Problems.** Find the requested integral of the vector function.

1. Let \( \mathbf{r}_1(t) = \frac{t}{1 + t^2} \mathbf{i} + \frac{1}{1 + t^2} \mathbf{j} + \frac{1}{1 - t^2} \mathbf{k} \). Find \( \int \mathbf{r}_1(t) dt \)

2. Let \( \mathbf{r}_2(t) = \sec 2t \mathbf{i} + \tan 3t \mathbf{j} + \ln(1 - t) \mathbf{k} \). Find \( \int \mathbf{r}_2(t) dt \)

3. Let \( \mathbf{r}_3(t) = (\sin(t), \sin(t) \cos(t), \sin^2(t)) \). Find \( \int_0^\pi \mathbf{r}_3(t) dt \).

Can you formulate a FTC II for vector functions?