Motivating Problem. For \( f(x, y) = x^3 \ln(y) \), find \( f_x, f_{xy}, f_y, \) and \( f_{yx} \).

Find the equation of the line tangent to the curve defined by the intersection of the surface \( f(x, y) = x^3 \ln y \) and the plane \( y = e \) at the point \((1, e, 1)\).

Find the equation of the line tangent to the curve defined by the intersection of the same surface \( f(x, y) = x^3 \ln y \) and the plane \( x = 1 \) at the point \((1, e, 1)\).

What do the contour lines (level curves) for \( f(x) = x^3 \ln y \) look like? What would the level curves look like if we zoomed in at a particular point—say \((1, e, 1)\)?

The tangent plane to \( f(x, y) = x^3 \ln(y) \) at the point \((1, e, 1)\) is the equation of the plane that contains both tangent lines above.
Suppose that $f$ has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

When you are asked to find the local linearization of function of 2-variables, they are simply asking for the equation of the tangent plane.

**Example.** Find the local linearization of $f(x, y) = x^2 + y^2$ at the point $(3, 4)$. Use it to estimate $f(2.9, 4.2)$ and $f(2, 2)$. Compare your answers to the true values.

$$f(x, y) \approx 25 + 6(x - 3) + 8(y - 4)$$
$$f(2.9, 4.2) \approx 26$$
$$f(2, 2) \approx 3$$

The **differential**, $df$ (or $dz$), at a point $(a, b)$ is the linear function of independent variables $dx$ and $dy$ given by the formula

$$df = f_x(a, b)dx + f_y(a, b)dy.$$  

The differential at a general point is often written as $df = f_x dx + f_y dy$.

Compute the differentials of the following functions:

- $f(x, y) = x^2 e^{5y}$
- $z = x \sin(xy)$
- $g(x, y) = x \cos(2x)$

Can you imagine what the differential for a function of three-variables might be? Take a stab at it for $f(x, y, z) = x^y z$. 

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Partial derivatives give the rate of change of a function in the directions parallel to the coordinate axes. What if we want to understand the rate of change as we travel in any direction?

We can always applied our original definition (looking at how the function changes between two points and seeking the limit as those points get closer and closer.)

**Theorem** The directional derivative of \( f \) at \((x_0, y_0)\) is the direction of a unit vector \( u = \langle a, b \rangle \) is

\[
D_u F(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
\]

if this limit exists.

**Example** Calculate the directional derivative of \( f(x, y) = x^2 + y^2 \) at \((1, 0)\) in the direction of the vector \( i + j \).

**Ans:** \( \sqrt{2} \)

Using the tangent plane as an approximation, we see how to calculate the directional derivative without a limit (\( u = \langle a, b \rangle \) is a unit vector):

\[
D_u f \approx \frac{df}{h} = \frac{fx dx + fy dy}{h} = \frac{fx ah + fy bh}{h} = f_x a + f_y b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle
\]
Definition. If \( f \) is a function of two variables \( x \) and \( y \), then the gradient of \( f \) is the vector function \( \nabla f \) defined by
\[
\nabla f = (f_x, f_y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.
\]

Problems. Use the gradient to find the directional derivation of \( f(x, y) = x + e^y \) at the point \((1, 1)\) in the direction of the vectors \( v_1, v_2, \) and \( v_3 \) given below.
\[
\begin{align*}
v_1 &= \langle 1, -1 \rangle \\
v_2 &= \mathbf{i} + 2\mathbf{j} \\
v_3 &= \langle 1, 3 \rangle
\end{align*}
\]

Questions.

1. In what direction does the largest directional derivative occur?
2. In what direction does no change in the directional derivative occur?
3. If \( f \) is the function that gives altitude above sea level for any point on the earth, where on our surface is the gradient \( \nabla f = \langle 0, 0 \rangle? \)