DIRECT CHARACTERIZATION OF SPECTRAL STABILITY
OF SMALL AMPLITUDE PERIODIC WAVES IN SCALAR
HAMILTONIAN PROBLEMS VIA DISPERSION RELATION *

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Abstract. The holy grail of the stability theory of nonlinear waves would be a formula that would allow to determine the stability or instability of a wave. We present results that form a step in this direction. The spectral stability of small-amplitude periodic waves in scalar Hamiltonian problems can be studied as a perturbation of the zero-amplitude case. A necessary condition for stability of the wave is that the unperturbed spectrum is restricted to the imaginary axis. Instability can come about through a Hamiltonian-Hopf bifurcation, i.e., of a collision of purely imaginary eigenvalues of the Floquet spectrum of opposite Krein signature. In recent work on the stability of small-amplitude waves the dispersion relation of the unperturbed problem was shown to play a central role. We demonstrate that the dispersion relation provides even more explicit information about wave stability: the sign of the product of the Krein signatures of two colliding eigenvalues to detect possible instabilities of non-zero amplitude waves turns is the root of an explicitly constructed polynomial of half the degree of the dispersion relation. Throughout we use balanced higher-order KdV equations as an example.

1. Introduction. We study the spectral stability of small-amplitude periodic traveling waves in scalar Hamiltonian partial differential equations:

\begin{equation}
    u_t = \partial_x \frac{\delta H}{\delta u}.
\end{equation}

Here

\[ u = u(x,t) = u(x + L,t), \quad x \in [0,L], \quad t > 0, \quad \text{and} \quad H = \int_0^L \mathcal{H}(u,u_x,\ldots) \, dx \]

is the Hamiltonian with density \( \mathcal{H} \). Without loss of generality, we let \( L = 2\pi \). This class of equations includes the KdV and modified KdV equation, the Kawahara equation, and other equations that arise in the study of dispersive problems in water waves, plasma physics etc [1, 11].

We assume that (1.1) has a trivial solution, i.e. \( \delta H/\delta u = 0 \) for \( u = 0 \), and \( H \) has an expansion \( H = H^0 + H^1 \), where \( H^0 \) is the quadratic part of \( H \) and \( H^1 \) contains the higher order terms:

\begin{equation}
    H^0 = -\frac{1}{2} \int_0^{2\pi} \sum_{j=0}^N \alpha_j \left( \partial_j^2 u \right)^2 .
\end{equation}
As a consequence, all linear terms in (1.1) are of odd degree, as even degree terms would introduce dissipation. We assume that \( N \) is a finite positive integer, and \( \alpha_j \in \mathbb{R} \). These assumptions exclude problems like the Whitham equation [6] for which \( N = \infty \), which remains a topic of investigation.

The now-standard approach to examine the stability of waves in Hamiltonian problems with symmetries is the theory developed by Vakhitov and Kolokolov [21] and Grillakis, Shatah, and Strauss [7, 8], which allows for the determination of spectral stability of waves of arbitrary amplitude. In that setup, spectral stability implies orbital (nonlinear) stability under certain conditions, emphasizing the importance of the spectral information of the linearized problem. Extensions of these results are found in [12, 14, 18, 19]. Periodic problems within the same framework were considered in [3, 9]. The use of any of these results relies on index theory requiring additional information about the PDE. That information is typically provided, for instance, by assuming something about the dimension of the kernel of the linearized problem. For small-amplitude waves extra information is often obtained through a perturbation of the zero-amplitude problem. We avoid index theory and study directly the collision of eigenvalues. The parallel work [20] illustrates how small-amplitude information is used to characterize the (in)stability of the waves. Here, we reduce the spectral stability problem for small-amplitude waves to the investigation of zeros of certain recurrently-defined polynomials, which appear in the theory of proper polynomial mappings [2, p172] and in orthogonal polynomial theory [17, Chapter 18]. To our knowledge, the connection between stability theory and these polynomials is new to the literature.

Our approach allows us to rigorously analyze the stability of the two-term balanced KdV-type equation, our case study. The results confirm the analytical and numerical predictions in [20]. Our method is closely related to the results in [6], where the spectrum of small-amplitude periodic solutions of Hamiltonian PDEs is determined directly from the dispersion relation of the PDE linearized about the zero solution. Our theory adds to the results in [6], and provides a simple and, importantly, a natural framework for studying the spectral stability of waves by perturbative methods.

The spectral stability of small-amplitude waves bifurcating from the trivial solution \( u = 0 \) at a critical velocity \( c = c_0 \) can be examined using regular perturbation theory of the spectrum of (1.1) linearized about \( u = 0 \) at \( c = c_0 \). We assume that \( u = 0 \) is spectrally stable, i.e., the spectrum of the linearized problem is restricted to the imaginary axis, since (1.1) is Hamiltonian.

In the periodic setting the whole spectrum of the zero-amplitude problem is needed. However, Floquet theory [13] allows to decompose the continuous spectrum to an infinite union of sets of discrete eigenvalues of eigenvalue problems parametrized by the Floquet multiplier \( \mu \). An important scenario for instability of small-amplitude waves on the bifurcation branch comes about through Hamiltonian-Hopf bifurcations [16, 22] producing symmetric pairs of eigenvalues off the imaginary axis, i.e., exponentially growing and therefore unstable modes. Such bifurcations require non-simple eigenvalues of the linearized problem at zero amplitude, i.e., “collided eigenvalues”. Furthermore, such colliding eigenvalues can split off from the imaginary axis only if they have opposite Krein signatures [16, 15].

Both the location of the eigenvalues and their Krein signatures are characterized by the dispersion relation of the linearized problem [6]. We show that even the collision of eigenvalues and the agreement of their signatures is directly characterized by the dispersion relation. This characterization is through the roots of a polynomial,
which is a reduction of the dispersion relation to a approximately half its degree. This is a surprising fact as it is by no means clear why such a characterization is possible, as the collisions of eigenvalues and their types are not itself objects that can be identified directly algebraically, particularly with a simpler algebraic relation than the eigenvalues themselves.

2. General Setting. We follow the steps outlined in Section III of [6]. Using a coordinate transformation \( x \to x - ct \) to a frame moving with the wave,

\[
\partial_t u = \partial_x (\frac{\delta H}{\delta u}) + c \partial_x u = \partial_x \left( \frac{\delta H}{\delta u} + cu \right) = \partial_x (\frac{\delta H_c}{\delta u}),
\]

where \( H_c \) is the modified Hamiltonian. The quadratic part of \( H_c \) is

\[
H_c^0 = \frac{c}{2} \int_0^{2\pi} u^2 \, dx - \frac{1}{2} \int_0^{2\pi} \sum_{j=0}^{N} \alpha_j (\partial_x u)^2 \, dx.
\]

Traveling wave solutions of (1.1) are stationary solutions \( U(x) \) of (2.1) and stationary points of \( H_c \).

2.1. Perturbation from the trivial state. Dispersion relation. For all \( c \in \mathbb{R} \), (2.1) has the trivial solution \( u(x,t) = 0 \). We linearize (2.1) about the zero solution to obtain an equation for the perturbation \( v = v(x,t) \) from the trivial state

\[
\partial_t v = c \partial_x v - \sum_{j=0}^{N} (-1)^j \alpha_j (\partial_x^{2j+1} v).
\]

We decompose \( v \) into a Fourier series in \( x \), \( v = \sum_{k=\infty}^{\infty} \exp(ikx) \hat{v}_k \), to obtain decoupled evolution equations for each of the Fourier coefficients \( \hat{v}_k = \hat{v}_k(t) \):

\[
\partial_t \hat{v}_k = -i \Omega(k) \hat{v}_k \quad k \in \mathbb{Z},
\]

where \( \Omega(k) \) given by

\[
\Omega(k) = \omega(k) - ck = \sum_{j=0}^{N} \eta_j k^{2j+1}, \quad \omega(k) = \sum_{j=0}^{N} \alpha_j k^{2j+1}, \quad \eta_j = \alpha_j - c \delta_{j1},
\]

is the dispersion relation of (2.3), obtained by letting \( v(x,t) = \exp(ikx - i\Omega t) \) in (2.3). Here \( \omega = \omega(k) \) is the dispersion relation in the original frame of reference corresponding to (1.1)–(1.2). Note that \( \omega(k) \) is an odd function.

2.2. Non-zero amplitude branches. Next, we discuss non-zero amplitude periodic solution branches of (2.1) bifurcating from the trivial state. A requirement for this is that a non-trivial stationary solution of (2.4) exists, i.e., \( \Omega(k) = 0 \), for \( k \in \mathbb{N}_0 \), since we have imposed that the solutions are \( 2\pi \) periodic. Thus

\[
c = c_k = \frac{\omega(k)}{k}, \quad k \in \mathbb{N}_0.
\]

For simplicity, we assume that a unique bifurcating branch emanates from \( c = c_k \). The solutions with \( k > 1 \) are \( 2\pi/k \) periodic. We focus on \( k = 1 \), i.e., \( c = \omega(1) \). The cases with \( k > 1 \) are treated analogously.
2.3. Floquet theory at zero amplitude. Using Floquet theory \[4, 13\] the spectral stability of the non-trivial solution \( U = U(x) \) of (2.1) on the bifurcation branch starting at \( c \) is determined by the growth rates of perturbations of the form

\[ v(x,t) = e^{\lambda t}V(x), \quad V(x) = e^{i\mu x} \sum_{n=-\infty}^{\infty} a_n e^{inx}. \]

Here \( \mu \in (-1/2, 1/2] \) is the Floquet exponent. Using (2.4) for the zero-amplitude case,

\[ \lambda = \lambda_n^{(\mu)} = -i\Omega(n + \mu) = -i\omega(n + \mu) + i(n + \mu)c, \quad n \in \mathbb{Z}. \]

The expression (2.8) is an explicit expression for the spectrum of the linearized stability problem for solutions of zero amplitude. Next, we examine how the spectrum of the linearization changes as the solution bifurcates away from zero amplitude.

2.4. Collisions of eigenvalues, Hamiltonian-Hopf bifurcations. After Floquet decomposition (2.7), the elements of the spectrum become eigenvalues of the \( \mu \)-parameterized operator obtained by replacing \( \partial_x \to \partial_x + i\mu \) in the linear stability problem. The eigenfunctions associated with these eigenvalues are (quasi)periodic and are bounded on the whole real line, see \[13, 5\] for details. For zero amplitude, the spectrum (2.8) is on the imaginary axis. Instabilities for small amplitude come about through collisions of purely imaginary eigenvalues at zero amplitude for a fixed value of \( \mu \). Away from the origin, eigenvalues generically split off from the axis through the Hamiltonian-Hopf bifurcations \[16, 22\] as the solution amplitude increases. Each such Hamiltonian-Hopf bifurcation produces a pair of eigenvalues off the imaginary axis that is symmetric with respect to the imaginary axis, thus yielding an exponentially growing eigenmode.

From (2.8), it is easy to detect eigenvalue collisions away from the origin. They correspond to solutions of \( \lambda_n^{(\mu)} = \lambda_m^{(\mu)} \neq 0, m, n \in \mathbb{Z}, m \neq n, \mu \in (-1/2, 1/2], \) i.e.,

\[ -i\Omega(n + \mu) = -i\omega(n + \mu) + i(n + \mu)c = -i\omega(m + \mu) + i(m + \mu)c = -i\Omega(m + \mu), \]

where \( c = c_1 \) is given by (2.6) with \( k = 1 \). Solving this equation results in values of \( \mu \) and \( n \) for which \( \lambda_n^{(\mu)} \) is an eigenvalue colliding with another one. Typically this is done by solving (2.9) for \( \mu \) for different fixed \( n \).

2.5. Krein signature. A necessary condition for two eigenvalues colliding on the imaginary axis to cause a Hamiltonian-Hopf bifurcation is that the eigenvalues have opposite Krein signatures. The Krein signature is the sign of the energy of the eigenmode associated with the eigenvalue. For a collision of eigenvalues to produce an instability this energy needs to be indefinite: a definite sign would entail bounded level sets of the energy, leading to perturbations remaining bounded.

For Hamiltonian systems with quadratic part given by (2.2) the eigenmode of the form \( v(x, t) = a_n \exp(i(n + \mu)x + \lambda_n^{(\mu)}t) + \text{c.c.} \), where c.c. stands for complex conjugate of the preceding term, contributes to \( H^0_e \) the relative energy (see [6])

\[ H^0_e(n, \mu) \sim -|a_n|^2 \frac{\Omega(n + \mu)}{n + \mu}. \]

Thus the Krein signature of \( \lambda_n^{(\mu)} \) is given by

\[ \kappa(\lambda_n^{(\mu)}) = -\text{sign} \left( \frac{\Omega(n + \mu)}{n + \mu} \right). \]
It follows that whether the signatures of two colliding eigenvalues $\lambda_n^{(\mu)}$ and $\lambda_m^{(\mu)}$ agree is characterized by the sign of the quantity
\[ q = q_{n,m}^{(\mu)} = \left( \frac{\Omega(n + \mu)}{n + \mu} \right) \left( \frac{\Omega(m + \mu)}{m + \mu} \right) = \frac{|\lambda_n^{(\mu)}|^2}{(n + \mu)(m + \mu)}. \]

We denote $Z = Z_{n,m}^{(\mu)} = (n + \mu)(m + \mu)$. Since $\lambda_n^{(\mu)} \neq 0$ we have
\[ \kappa(\lambda_n^{(\mu)})\kappa(\lambda_m^{(\mu)}) = \text{sign}(q) = \text{sign}((n + \mu)(m + \mu)) = \text{sign}(Z). \]
Therefore the agreement of Krein signatures of coinciding eigenvalues away from the origin is characterized by the sign of $Z$.

By shifting $\mu$ and denoting $\tilde{\mu} = m + \mu$ and $\tilde{n} = n - m$ the collision condition (2.9) becomes $\Omega(\tilde{n} + \tilde{\mu}) = \Omega(\tilde{\mu})$. Dropping tildes,
\[ (2.10) \quad \Omega(n + \mu) = \Omega(\mu), \]
with the associated quantity $Z = \mu(n + \mu)$.

### 3. Recurrent Sequences of Polynomials.

Before we revisit (2.10) in the next section, we need some results on two particular sequences of polynomials.

**Lemma 3.1.** Let $a, b \in \mathbb{C}$, $m \in \mathbb{N}_0$, and
\[ t_m = a^m + (-b)^m. \]
Then
\[ t_{m+1} = (a - b)t_m + (ab)t_{m-1}, \quad m \geq 1. \]

**Proof.**
\[ t_{m+1} = (a - b)(a^m + (-1)^mb^m) + ab(a^{m-1} + (-1)^{m-1}b^{m-1}) = (a - b)t_m + (ab)t_{m-1}. \]

Since $t_0 = 2$ and $t_1 = a - b$, by induction all $t_m$ can be written as polynomials in the two variables $a - b$ and $ab$. $t_m = t_m(a - b, ab)$. Further, $t_m$ is a homogeneous polynomial in $a$ and $b$ of degree $m$. We introduce $s_m$ by $t_m = (a - b)^m s_m(\gamma)$, i.e.,
\[ (3.1) \quad s_m = s_m(\gamma) := \frac{t_m(a - b, ab)}{(a - b)^m}, \quad \text{with } \gamma := \frac{ab}{(a - b)^2}. \]
The sequence $s_m$ is characterized recursively by
\[ (3.2) \quad s_{m+1} = s_m + \gamma s_{m-1}, \quad m \geq 1, \quad s_0 = 2, \quad s_1 = 1, \]
which shows that $s_m$ is polynomial in $\gamma$ of degree $m/2$ ($m$ even) or $(m - 1)/2$ ($m$ odd). Solving the recurrence relationship,
\[ (3.3) \quad s_m(\gamma) = \psi^+_m + \psi^-_m, \quad m \geq 0, \quad \psi_{\pm} := \frac{1}{2} \left( 1 \pm \sqrt{1 + 4\gamma} \right). \]
Note that these imply that
\[ (3.4) \quad s_m(0) = 1, \quad s_m(-1/4) = 2^{1-m}. \]

A few lemmas are required for the following sections. The reader may choose to go directly to the next section and revisit these results as needed.
Lemma 3.2. Let $s_m(\gamma)$ be as above. Then

\begin{align*}
(3.5) & \quad s_m(\gamma) \geq 2^{-m} - (m-1), \quad \text{for all } \gamma \geq -1/4 \text{ and } m \geq 0, \\
(3.6) & \quad s_m(\gamma) < 1, \quad \text{for all } \gamma \in [-1/4, 0) \text{ and } m \geq 2, \\
(3.7) & \quad s_m(\gamma) > 1, \quad \text{for all } \gamma > 0 \text{ and } m \geq 2.
\end{align*}

Proof. First, for $\gamma \geq -1/4$, $s_m(\gamma)$ is an increasing function of $\gamma$ since $s'_m(\gamma) = \gamma/(1 + 4\gamma) - \gamma/(1 + 1/4) > 0$. The inequality (3.5) follows from this and $s_m(-1/4) = 2^{-m} - (m-1)$. Equation (3.6) follows from the fact that $\psi_+ \in (0, 1)$ for $\gamma \in [-1/4, 0)$. Hence $s_{m+1}(\gamma) < s_m(\gamma)$ for all $m \geq 0$. Then $s_1(\gamma) = 1$ yields the claim.

Finally, we prove (3.7). For $m = 2$ and $m = 3$, $s_2(\gamma) = 1 + 2\gamma > 1$, and $s_3(\gamma) = 1 + 3\gamma > 1$ for $\gamma > 0$. Then (3.7) follows directly from (3.2).

Lemma 3.3. For all $m \geq 0$ and $\gamma \geq -1/4$,

\begin{align*}
(3.8) & \quad s_{m+2}(\gamma) \geq -\gamma s_m(\gamma), \\
(3.9) & \quad s_{m+1}(\gamma) \geq s_m(\gamma)/2, \\
(3.10) & \quad s_{m+1}(\gamma) \leq [1 + m(1 + 4\gamma)] s_m(\gamma)/2.
\end{align*}

Proof. The inequality (3.8) is equivalent to $s_{m+2} - s_{m+1} + s_m(\gamma) \geq 0$. Using the recurrence relation (3.2), it reduces to $2s_{m+2} - s_{m+1} \geq 0$, i.e., $2s_{m+2} \geq s_{m+1}$, $m \geq 0$. Thus (3.8) and (3.9) are equivalent except for (3.9) with $m = 0$, which is trivially satisfied ($2s_1 = 2 = s_0$). Also note that $s_m(\gamma) \geq 0$ for $m \geq 0$ and $\gamma \geq 0$ and (3.8) is satisfied for $\gamma \geq 0$. In the rest of the proof of (3.8), we assume that $m \geq 1$ and $\gamma \in [-1/4, 0)$. We shift $m \to m + 1$ in (3.9). $m \geq 0$, which becomes

\begin{equation}
(3.11) \quad \left(\psi_+ - \frac{1}{2}\right) \psi_+^{m+1} + \left(\psi_- - \frac{1}{2}\right) \psi_-^{m+1} \geq 0.
\end{equation}

Since $\psi_- = 1 - \psi_+$ for $\gamma \in [-1/4, 0)$, (3.11) is equivalent to

\begin{equation}
\left(\psi_+ - \frac{1}{2}\right) \left[\psi_+^{m+1} - \psi_-^{m+1}\right] \geq 0,
\end{equation}

which is satisfied for $\gamma \in [-1/4, 0)$ since $\psi_+ \geq 1/2$ and $\psi_+ > \psi_-$. This proves (3.9) and (3.8).

We turn to (3.10). Note that (3.10) holds for $m = 0$. For $m \geq 1$, first we consider $\gamma \geq 0$. Using (3.2),

\begin{equation}
2(s_m + \gamma s_{m-1}) \leq [m(1 + 4\gamma) + 1] s_m,
\end{equation}

i.e.,

\begin{equation}
2\gamma s_{m-1} \leq [m(1 + 4\gamma) - 1] s_m = (m - 1)s_m + 4m\gamma s_m.
\end{equation}

But $m \geq 1$ and $s_m \geq 0$. Therefore $(m - 1)s_m \geq 0$ and (3.12) follows from $2\gamma s_{m-1} \leq 4m\gamma s_m$, i.e., $s_m \geq s_{m-1}/2m$, which holds, according to (3.9).

Next, consider $\gamma \in [-1/4, 0)$. We write (3.10) as $2s_{m+1} - s_m \leq m(1 + 4\gamma)s_m$, and use (3.3) to obtain

\begin{equation}
\psi_+^m \left(\psi_+ - \frac{1}{2}\right) + \psi_-^m \left(\psi_- - \frac{1}{2}\right) \leq \frac{m(1 + 4\gamma)}{2} \left(\psi_+^m + \psi_-^m\right).
\end{equation}
Using \( \psi_+ + \psi_- = 1 \),
\[
\left( \psi_+ - \frac{1}{2} \right) (\psi_+^m - \psi_-^m) \leq \frac{m(1 + 4\gamma)}{2} (\psi_+^m + \psi_-^m).
\]
Since
\[
\psi_+ - \frac{1}{2} = \frac{\sqrt{1 + 4\gamma}}{2},
\]
Equation (3.10) is equivalent to
\[
(\psi_+^m - \psi_-^m) \leq m\sqrt{1 + 4\gamma} (\psi_+^m + \psi_-^m),
\]
or
\[
(3.13)
\]
\[
\psi_+^m \left( 1 - m\sqrt{1 + 4\gamma} \right) \leq \psi_-^m \left( 1 + m\sqrt{1 + 4\gamma} \right).
\]
Both \( \psi_+ \) and \( 1 + m\sqrt{1 + 4\gamma} \) are positive, and
\[
\frac{\psi_-}{\psi_+} = 1 - \frac{\sqrt{1 + 4\gamma}}{1 + \sqrt{1 + 4\gamma}} = \frac{1 + 2\gamma - \sqrt{1 + 4\gamma}}{-2\gamma}.
\]
It follows that proving (3.13) is equivalent to proving
\[
(3.14)
\]
\[
\frac{1 - m\sqrt{1 + 4\gamma}}{1 + m\sqrt{1 + 4\gamma}} \leq \left( \frac{1 + 2\gamma - \sqrt{1 + 4\gamma}}{-2\gamma} \right)^m.
\]
We prove (3.14) by induction for \( m \geq 0 \). For \( m = 0 \), (3.14) is trivially satisfied. Assume that (3.14) holds for \( m \). Using this, we have to show that (3.14) holds for \( m + 1 \). This amounts to showing that
\[
(3.15)
\]
\[
\frac{1 - m\sqrt{1 + 4\gamma}}{1 + m\sqrt{1 + 4\gamma}} \leq \frac{1 - (m + 1)\sqrt{1 + 4\gamma}}{1 + (m + 1)\sqrt{1 + 4\gamma}}.
\]
Multiplying (3.15) by all (positive) denominators simplifies to an inequality which holds for all \( \gamma \in [-1/4, 0] \):
\[
m(m + 1)(1 + 4\gamma)^{3/2} \left( 1 - \sqrt{1 + 4\gamma} \right) \geq 0.
\]

**Lemma 3.4.** For all \( m \geq 2 \),
\[
\begin{align*}
\gamma(2^m - 1)s_{m-1}(\gamma) + s_{m+1}(\gamma) & \geq 1, \text{ for } \gamma \in [-1/4, 0]. \quad (3.16) \\
\gamma(2^m - 1)s_{m-1}(\gamma) + s_{m+1}(\gamma) & \leq 1, \text{ for } \gamma \geq 0. \quad (3.17)
\end{align*}
\]

**Proof.** We prove (3.16) using induction. For \( m = 2 \) and \( m = 3 \)
\[
\begin{align*}
-\gamma(2^2 - 1)s_1(\gamma) + s_3(\gamma) &= -3\gamma + 1 + 3\gamma = 1, \\
-\gamma(2^3 - 1)s_2(\gamma) + s_4(\gamma) &= 1 - 3\gamma(1 + 4\gamma) \geq 1.
\end{align*}
\]
Assume (3.16) holds for some \( m \geq 3 \), i.e.,
\[
(3.18) \quad -\gamma(2^m - 1)s_{m-1} + s_{m+1} \geq 1.
\]
By Lemma 3.3, $s_m + \gamma s_{m-2} \geq 0$. Using (3.2) this becomes $s_{m-1} + 2\gamma s_{m-2} \geq 0$. After multiplication by $2^m - 1 > 0$, we obtain the equivalent form

$$(2^m - 1)s_{m-1} + 2\gamma(2^m - 1)s_{m-2} = (2^m - 1)s_{m-1} + \gamma(2^{m+1} - 2)s_{m-2} \geq 0,$$

which, using (3.2), is rewritten as

$$(3.19) \quad 2^m s_{m-1} + \gamma(2^{m+1} - 1)s_{m-2} - s_m \geq 0.\quad \tag{3.19}$$

Multiplying (3.19) by $-\gamma \geq 0$ and adding (3.18) gives

$$-\gamma(2^m - 1)(s_{m-1} + \gamma s_{m-2}) + (s_{m+1} + \gamma s_m) \geq 1,$$

which is rewritten as

$$-\gamma(2^m - 1)s_m + s_{m+2} \geq 1.$$

This concludes the proof of the second induction step.

Next we prove (3.17). The statement is true for $m = 2$ and $m = 3$:

$$-\gamma(2^2 - 1)s_1(\gamma) + s_3(\gamma) = 1, \quad -\gamma(2^3 - 1)s_2(\gamma) + s_4(\gamma) = 1 - 3\gamma - 12\gamma^2 \leq 1.$$

Assume (3.17) holds for some $m \geq 3$, i.e.,

$$-\gamma(2^m - 1)s_{m-1} + s_{m+1} \leq 1.\quad \tag{3.20}$$

By Lemma 3.3, $s_m + \gamma s_{m-2} \geq 0$ or equivalently $s_{m-1} + 2\gamma s_{m-2} \geq 0$, so that

$$(2^m - 1)s_{m-1} + 2\gamma(2^m - 1)s_{m-2} = (2^m - 1)s_{m-1} + \gamma(2^{m+1} - 2)s_{m-2} \geq 0.$$

This is rewritten as

$$2^m s_{m-1} + \gamma(2^{m+1} - 1)s_{m-2} - s_m \geq 0.$$

We reverse this inequality by multiplying it by $-\gamma \leq 0$, and add (3.20) to it to obtain

$$-\gamma(2^m - 1)s_{m-1} + s_{m+1} - \gamma 2^m s_{m-1} - \gamma(2^{m+1} - 1)\gamma s_{m-2} + \gamma s_m \leq 1,$$

which reduces to

$$-\gamma(2^{m+1} - 1)s_m + s_{m+2} \leq 1.$$

This concludes the proof of the second induction step. \qed

**Lemma 3.5.** The sequence

$$\frac{2^m s_{m+1}(\gamma) - 1}{2^m - 1}, \quad m \geq 1,$$

is nonincreasing in $m$ for $\gamma \in [-1/4, 0]$.

**Proof.** We prove that for $m \geq 1$, \[
\frac{2^m s_{m+1} - 1}{2^m - 1} \geq \frac{2^{m+1} s_{m+2} - 1}{2^{m+1} - 1}.
\]
Using the recurrence relation (3.2), this is equivalent to

$$s_{m+1} \geq \gamma (2^{m+1} - 2)s_m + 1 \iff s_{m+2} - \gamma (2^{m+1} - 1)s_m \geq 1,$$

which follows directly from Lemma 3.4. \qed
Lemma 3.6. The sequence
\[
\frac{2^m s_{m+1}(\gamma) - 1}{m(m+1)}, \quad m \geq 1,
\]
is nondecreasing in \( m \) for \( \gamma \geq -1/4 \).

Proof. We use induction to show that for \( m \geq 1 \)
\[
\frac{2^m s_{m+1} - 1}{m(m+1)} \leq \frac{2^{m+1} s_{m+2} - 1}{(m+1)(m+2)},
\]
or equivalently, for \( m \geq 1 \),
\[
(m+2)2^m s_{m+1} \leq m2^{m+1} s_{m+2} + 2. \tag{3.21}
\]
The inequality (3.21) holds for \( m = 1 \) as
\[
6s_2 = 6(1+2\gamma) = 4(1+3\gamma) + 2 = 4s_3 + 2.
\]
Using (3.2) to expand \( s_{m+2} \) in (3.21) we obtain
\[
(m+2)2^m s_{m+1} \leq m2^{m+1}(s_{m+1} + \gamma s_m) + 2,
\]
and (3.21) is equivalent to
\[
2^m s_{m+1} - \gamma m2^{m+1} s_m \leq (m-1)2^m s_{m+1} + 2.
\]
It suffices to prove that
\[
2^m s_{m+1} - \gamma m2^{m+1} s_m \leq (m+1)2^{m-1} s_m, \tag{3.22}
\]
since the induction assumption (3.21) for \( m \to m-1 \) implies
\[
(m+1)2^{m-1} s_m \leq (m-1)2^m s_{m+1} + 2.
\]
But (3.22) follows directly from (3.10) of Lemma 3.3 as it is equivalent to
\[
s_{m+1} \leq [1 + m(1+4\gamma)] s_m.
\]
Finally, we prove two lemmas that provide bounds for growth of the sequence \( \{s_m(\gamma) - 1\} \).

Lemma 3.7. The sequence
\[
(s_m(\gamma) - 1)/m, \quad m \geq 3,
\]
is nondecreasing in \( m \) for \( \gamma \in [-1/4, 0) \).

Proof. The statement is equivalent to \((m+1)s_m \leq ms_{m+1} + 1\), which we prove by induction. First, for \( m = 3 \) we have
\[
4s_3 < 3s_4 + 1, \quad \text{i.e.,} \quad 4(1+3\gamma) < 3(1+4\gamma+2\gamma^2) + 1 \quad \text{which holds for} \ \gamma \neq 0.
\]
Assume that the statement holds for \( m \to m-1 \), i.e., \( ms_{m-1} \leq (m-1)s_m + 1 \), which is equivalent to \( s_m \leq m(s_m - s_{m-1}) + 1 \). Thus \( s_m \leq m\gamma s_{m-2} + 1 \). However, for \( \gamma \in [-1/4, 0) \) and \( m \geq 2 \) one has \( 0 < s_{m-1} < s_{m-2} \) and thus \( s_m \leq m\gamma s_{m-1} + 1 \). The claim follows by an application of (3.2) to \( s_{m-1} \).

Lemma 3.8. The sequence
\[
(s_{m+1}(\gamma) - 1)/(2^{-m} - 1), \quad m \geq 1,
\]
is (i) nondecreasing in \( m \), for \( \gamma \in [-1/4, 0) \); (ii) nonincreasing in \( m \), for \( \gamma > 0 \).
Proof. First, we prove (i), which is equivalent to \((2^{m+1} - 2)s_{m+2} + 1 \leq (2^{m+1} - 1)s_{m+1}\). Using (3.2) in the form \(s_{m+2} = s_{m+1} + \gamma s_m\), this reduces to \(s_{m+1} - 2\gamma(2^m - 1)s_m \geq 1\). This follows directly from a combination of \(-\gamma(2^m - 1)s_{m-1} + s_{m+1} \geq 1\), which holds for all \(m \geq 2\), and \(\gamma \in [-1/4, 0)\) by Lemma 3.4 and \(s_{m-1} \leq 2s_m\) (see (3.9)).

Next we prove (ii) by an analogous argument. We have to show that \((2^{m+1} - 2)s_{m+2} + 1 \geq (2^{m+1} - 1)s_{m+1}\), which reduces (by (3.2) in the form \(s_{m+2} = s_{m+1} + \gamma s_m\)) to \(s_{m+1} - 2\gamma(2^m - 1)s_m \leq 1\). This follows from \(-\gamma(2^m - 1)s_{m-1} + s_{m+1} \leq 1\) (by Lemma 3.4) and \(s_{m-1} \leq 2s_m\) (by (3.9)) for all \(m \geq 2\).


We prove that for scalar Hamiltonian problems (2.1)–(2.2) of order \(2N+1\), the polynomial equation (2.10) characterizing the collision of eigenvalues at zero-amplitude resulting in Hamiltonian-Hopf bifurcations, and thus instability of non-zero amplitude periodic waves, can be written in the form \(Q(Z, n) = 0\), where \(Q\) is polynomial in both of its variables, of degree \(N\).

**Theorem 4.1.** Let \(\Omega = \Omega(k)\) be a complex-valued odd polynomial of degree \(2N+1\). Then

\[
\Omega(n + \mu) - \Omega(\mu) = Q(Z, n)
\]

where \(Q(Z, n)\) is a polynomial of degree \(N\) in \(Z = \mu(\mu + n)\) and \(n\) with complex coefficients.

**Proof.** With \(a := n + \mu, b := \mu, a - b = n\), we have (see (3.1)):

\[
Z = \mu(\mu + n) = n^2\gamma, \quad \text{i.e.}, \quad \gamma = \frac{Z}{n^2},
\]

and (4.1) becomes

\[
\Omega(a) - \Omega(b) = Q(ab, a - b).
\]

Using (2.5), \(\Omega(k) = \sum_{j=0}^{N} \eta_j k^{2j+1}\). Then

\[
\Omega(a) - \Omega(b) = \sum_{j=0}^{N} \eta_j (a^{2j+1} - b^{2j+1}) = \sum_{j=0}^{N} \eta_j t_{2j+1}(a - b, ab) = \sum_{j=0}^{N} \eta_j n^{2j+1} s_{2j+1}(\gamma),
\]

where we use the results of the previous section. The right-hand side is a polynomial of degree \(N\) in \(\gamma\).

In the proof of the theorem, we have derived a formula for the value of \(Z\) characterizing the sign of the Krein signatures of two coinciding eigenvalues:

\[
0 = \sum_{j=0}^{N} \eta_j n^{2j+1} s_{2j+1}(\gamma).
\]

As before, this equation is solved for \(\gamma\) for different fixed values of \(n\). After solving for \(\gamma\), it is necessary to check that \(\gamma\) gives rise to a real value of \(\mu\) by solving the quadratic equation with the unknown \(\mu\):

\[
\mu(\mu + n) = c = \gamma n^2.
\]
Thus

\[(4.5) \quad \mu_{1,2} = \frac{-n \pm \sqrt{n^2 + 4\gamma n^2}}{2} = \frac{n}{2} \left( -1 \pm \sqrt{1 + 4\gamma} \right).\]

We are interested in negative values of \(\gamma\) (possible coincidence of two eigenvalues of opposite signature). Then any root \(\gamma \in [-1/4, 0]\) corresponds to a collision of two eigenvalues of opposite signature. If \(\gamma < -1/4\), \(\gamma\) does not correspond to a collision of two purely imaginary eigenvalues as \(\mu\) is not real. If \(\gamma > 0\) then there is a collision of two eigenvalues of the same signature. We have proved the following main theorem characterizing the spectral stability of small-amplitude traveling waves of (1.1).

**Theorem 4.2.** Consider a scalar 2\(\pi\)-periodic Hamiltonian partial differential equation of the form (1.1) and assume that \(u = 0\) is a spectrally stable solution. Let (2.5) be the dispersion relation of the equation linearized about \(u = 0\) in a reference frame moving with the velocity \(c\). Then a branch of travelling wave solutions of (1.1) with velocity \(c\) bifurcates from the trivial solution at \(c = \omega(1)\), see (2.6). A necessary condition for a Hamiltonian-Hopf bifurcation at zero-amplitude characterizing a loss of spectral stability of small amplitude solutions on the bifurcating branch is that (4.4) has a root \(\gamma, \gamma \in [-1/4, 0]\).

5. Balanced Higher Order KdV equations. We demonstrate the power of Theorem 4.2 by explicitly characterizing the stability regions for small-amplitude periodic solutions of KdV-type equations with two linear terms of odd order:

\[(5.1) \quad u_t = \partial_x f(u) + A \partial_x^{2q+1} u + B \partial_x^{2p+1} u,\]

subject to periodic boundary conditions. Here \(p > q\) are positive integers, \(A, B \in \mathbb{R}\) are non-zero coefficients, and \(f(u)\) is a smooth function of \(u\) and its spatial derivatives with \(f(0) = 0\), containing no linear terms. The literature on this topic is limited. Most relevant is [10], where \(f(u) \sim u^2\) (the Kawahara equation), and the period of the solutions is not fixed. It is concluded there that for solutions for which the amplitude scales as the 1.25-th power of the speed, solutions are spectrally stable. No conclusion is obtained for other solutions. Our investigation does not require this scaling, nor does it restrict the type of nonlinearity. Also relevant is [3], where the typical stability approach of [9] is extended to systems with singular Poisson operator like (1.1), but the theory is not applied to (5.1). A mostly numerical investigation of equations like (5.1) is undertaken in [20]. As stated, our theory builds almost exclusively on [6] and our rigorous results agree with numerical results in [20] where the special case \(p = 2, q = 1, A, B > 0\) was considered.

Traveling wave solutions \(u = U(x - ct)\) with wave velocity \(c\) satisfy

\[-cU' = \partial_x f(U) + AU^{2q+1} + BU^{2p+1}.\]

The spectral stability of small-amplitude waves that bifurcate at zero amplitude from the trivial solution \(U = 0\) is characterized by the growth of the solutions of the linear equation

\[(5.2) \quad v_t = cv_x + Av^{2q+1} + Bv^{2p+1},\]

with dispersion relation

\[\Omega = \Omega_{p,q}(k) = -ck - A(-1)^q k^{2q+1} - B(-1)^p k^{2p+1} = -ck - \alpha k^{2q+1} + \beta k^{2p+1}.\]
where we have introduced
\[ (5.3) \quad \alpha = A(-1)^q, \quad \beta = -B(-1)^p. \]

Without loss of generality, we assume that \( \alpha > 0 \). If not, the transformation \( x \to -x \) (i.e., \( k \to -k \)), and \( c \to -c \) can be used to switch the sign of \( \alpha \). The scaling symmetry of the equation allows us to equate \( \alpha = 1 \) hereafter. The choice of opposite signs in front of \( \alpha \) and \( \beta \) in (5.3) is intuitive: if \( \alpha \) and \( \beta \) have opposite sign the Hamiltonian energy (2.2) is definite and all eigenvalues have the same signature. This rules out Hamiltonian-Hopf bifurcations and the spectral instabilities following from them. In other words, the interesting case for our considerations is that both \( \alpha \) and \( \beta \) are positive. Lastly, since we study bifurcations from the first Fourier mode \( k = 1 \), \( c = \beta - \alpha = \beta - 1 \).

According to Theorem 4.1, eigenvalue collisions at zero-amplitude are characterized by the roots \( \gamma \) of
\[ \gamma \in \left[ -\frac{1}{4}, 0 \right). \]

Our goal is to find the parameter range \( (\beta, n) \) for which the root \( \gamma \) of (5.4) satisfies \( \gamma = 0 \). An important role is played by the interval end points \( \gamma = 0 \) and \( \gamma = -1/4 \). By (3.4) for \( \gamma = 0 \) we have
\[ \beta(n^2 - 1) - (n^2 - 1) = 0 \]
and therefore we set
\[ \beta_0 = \beta_0(n) = \frac{n^2q - 1}{n^2p - 1}. \]

On the other hand (5.4) reduces for \( \gamma = -1/4 \) by (3.4) to
\[ (5.5) \quad \beta_{-1/4} = \beta_{-1/4}(n) = \left\{ \left( \frac{n}{2} \right)^2q - 1 \right\} \left/ \left\{ \left( \frac{n}{2} \right)^2p - 1 \right\} \right. \]

We need the following two lemmas.

**Lemma 5.1.** Let \( \alpha > 0 \). The function
\[ g(x) = \frac{x\alpha^x}{\alpha^x - 1} \]
is increasing on \((0, \infty)\).

**Proof.** The condition \( g'(x) > 0 \) is equivalent to \( \alpha^x = e^{x \ln \alpha} > 1 + x \ln \alpha \). This follows directly from the Taylor expansion of \( e^x \) at \( x = 0 \) with equality reached for \( x = 0 \). \( \square \)

**Lemma 5.2.** Let \( a > b > 0 \). Define
\[ f(n) = f_{a,b}(n) = \frac{n^{a-b} - 1}{n^a - 1}. \]

We define \( f(1) = \lim_{n \to 1} f(n) = (a - b)/a \). Then \( f(n) \) is a decreasing function on \([1, \infty)\).
Proof. The inequality \( f'(n) < 0 \) is equivalent to \( a(n^b - 1) < b(n^a - 1) \), i.e.,
\[
\frac{a}{b} < \frac{n^a - 1}{n^b - 1}.
\]
The estimate (5.6) for \( n > 1 \) follows from the fact that the function
\[
h(n) = \frac{n^a - 1}{n^b - 1}, \quad a > b > 0,
\]
is increasing on \([1, \infty)\), where \( h(1) = \lim_{n \to 1} h(n) = a/b\). The inequality \( h'(n) > 0 \)
reduces to
\[
\frac{an^a}{n^a - 1} > \frac{bn^b}{n^b - 1},
\]
which holds for \( a > b > 0 \) and \( n > 1 \) by Lemma 5.1. Lemma 5.2 follows by continuity of \( h(n) \) at \( n = 1 \).

It follows immediately that for \( n \geq 3 \), \( \beta_0 < \beta_{-1/4} \), since this inequality may be rewritten as \( f_{2p,2q}(n) < f_{2p,2q}(2) \).

### 5.1. Collisions of eigenvalues of opposite signature
Since the thresholds \( \gamma = 0 \) and \( \gamma = -1/4 \) correspond, respectively, to \( \beta = \beta_0(n) \) and \( \beta = \beta_{-1/4}(n) \), where \( \beta_0(n) < \beta_{-1/4}(n) \), one may conjecture (for \( n \geq 3 \), since for \( n = 1,2 \) either \( \beta_0 \) or \( \beta_{-1/4} \) is not defined) that collisions of eigenvalues of opposite Krein signature happen for \( \beta \in (\beta_0(n), \beta_{-1/4}(n)) \). For \( \beta < \beta_0(n) \) one expects collisions of eigenvalues of the same signature and finally for \( \beta > \beta_{-1/4}(n) \) one expects no collisions as the roots \( \mu \) of (4.5) are not real (see Fig. 5.1). As we prove next, this is true. The cases \( n = 1 \) and \( n = 2 \) are treated separately.

![Parameter regimes for \( \beta \), \( \beta \leq \beta_0(n) \), \( \beta \in (\beta_0(n), \beta_{-1/4}(n)) \), and \( \beta > \beta_{-1/4}(n) \).](image)

#### Theorem 5.3. Case \( n \geq 3 \)

Let \( p, q \), \( p > q \), be positive integers and let \( n \in \mathbb{N} \), \( n \geq 3 \). The presence and character of collisions of eigenvalues of the linearized problem (5.2) at zero amplitude at \( c = c_1 = \beta - \alpha \) depends on the Fourier mode \( n \) of the perturbation in the following way:

(i) If \( n \) is such that \( \beta < \beta_0(n) \), then there is a collision of eigenvalues of the same signature, i.e., there is a root of (5.4) with \( \gamma > 0 \) and there is no root with \( \gamma \in [-1/4, 0) \);

(ii) If \( n \) is such that \( \beta_0(n) < \beta \leq \beta_{-1/4}(n) \), then there is a collision of eigenvalues of opposite signature, i.e., there is a root \( \gamma \) of (5.4) such that \( \gamma \in [-1/4, 0) \);

(iii) If \( n \) is such that \( \beta_{-1/4}(n) < \beta \), then there is no collision of eigenvalues, i.e., all roots \( \gamma \) of (5.4) satisfy \( \gamma < -1/4 \).

Proof. Part (ii). We show that for all \( n \geq 3 \) and \( \beta_0(n) < \beta \leq \beta_{-1/4}(n) \) there exists \( \gamma \in [-1/4, 0) \) satisfying \( R(\gamma) = 0 \). Therefore by (5.4), in such a parameter regime there is a collision of eigenvalues of opposite Krein signature.

It is easy to see that
\[
R(0) = \beta(n^{2p} - 1) - (n^{2q} - 1) > \beta_0(n^{2p} - 1) - (n^{2q} - 1) = 0,
\]
and,

$$ R(-1/4) = \beta \left( \frac{n^{2p}}{2^{2p}} - 1 \right) - \left( \frac{n^{2q}}{2^{2q}} - 1 \right) \leq \beta_{-1/4} \left( \frac{n^{2p}}{2^{2p}} - 1 \right) - \left( \frac{n^{2q}}{2^{2q}} - 1 \right) = 0. $$

Thus $R(0) > 0 \geq R(-1/4)$ and the polynomial $R(\gamma)$ has a real root $\gamma \in [-1/4, 0)$.

**Part (i).** Since $\beta < \beta_0(n) < \beta_{-1/4}(n)$ the same argument as in Part (ii) yields $R(-1/4) < 0$. Also,

$$ R(0) = \beta(n^{2p} - 1) - (n^{2q} - 1) < \beta_0(n^{2p} - 1) - (n^{2q} - 1) = 0. $$

We prove that $R(\gamma) = \beta(n^{2p}s_{2p+1}(\gamma) - 1) - (n^{2q}s_{2q+1}(\gamma) - 1) < 0$ for all $\gamma \in [-1/4, 0]$. By Lemma 3.2 for $n \geq 3$ and $p \geq 1$,

$$ n^{2p}s_{2p+1}(\gamma) \geq \frac{2^{2p}}{2^{2p+1}} > 1. $$

Thus for all $\gamma \in [-1/4, 0]$ and $\beta < \beta_0$,

$$ R(\gamma) = \beta(n^{2p}s_{2p+1}(\gamma) - 1) - (n^{2q}s_{2q+1}(\gamma) - 1) $$

$$ < \beta_0(n^{2p}s_{2p+1}(\gamma) - 1) - (n^{2q}s_{2q+1}(\gamma) - 1). $$

We prove that the right-hand side of (5.7) is non-positive. This is equivalent to

$$ \beta_0(n) = \frac{n^{2q} - 1}{n^{2p} - 1} \leq \frac{n^{2q}s_{2q+1}(\gamma) - 1}{n^{2p}s_{2p+1}(\gamma) - 1}, $$

or to

$$ s_{2q+1} \geq s_{2p+1}[1 - \theta(n)] + \theta(n), \quad \text{where } \theta(n) := \frac{n^{2p} - n^{2q}}{n^{2p+2q} - n^{2q}}. $$

Clearly $0 < \theta(n) < 1$. Since $s_{2p+1} < 1$ it suffices to prove (5.9) for the $n$ that maximizes $\theta(n)$, $n \geq 2$. However, by Lemma 5.2 for $p > q \geq 1$, $\max_{n \geq 2} \theta(n) = \theta(2)$ and it suffices to prove $s_{2q+1} \geq s_{2p+1}(1 - \theta(2)) + \theta(2)$, i.e.,

$$ s_{2q+1}2^{2q}(2^{2p} - 1) \geq s_{2p+1}2^{2p}(2^{2q} - 1) + 2^{2p} - 2^{2q}. $$

Therefore (5.8) follows directly from Lemma 3.5 as it is equivalent for $p > q \geq 1$ to

$$ \frac{2^{2q}s_{2q+1} - 1}{2^{2q} - 1} \geq \frac{2^{2p}s_{2p+1} - 1}{2^{2p} - 1}. $$

Hence we proved $R(\gamma) < 0$ for all $\gamma \in [-1/4, 0]$. On the other hand $R(\gamma)$ is an even order polynomial with a positive leading coefficient, i.e., $R(\gamma) \rightarrow \infty$ as $\gamma \rightarrow \infty$. Therefore there exists $\gamma_0 > 0$ such that $R(\gamma_0) = 0$. Such a root corresponds by (4.2) to $c > 0$ and thus by (4.5) it corresponds to a real value of $\mu$. Therefore in this regime there is a collision of two eigenvalues of the same signature.

**Part (iii).** Note that $R(0) > 0$. We show that $R(\gamma) > 0$ for $\gamma \geq -1/4$. First,

$$ R(-1/4) = \beta \left( \frac{n^{2p}}{2^{2p}} - 1 \right) - \left( \frac{n^{2q}}{2^{2q}} - 1 \right) > \beta_{-1/4} \left( \frac{n^{2p}}{2^{2p}} - 1 \right) - \left( \frac{n^{2q}}{2^{2q}} - 1 \right) = 0. $$
For $\gamma \geq -1/4$, 
\begin{equation}
R(\gamma) = \beta (n^{2p}s_{2p+1}(\gamma) - 1) - (n^{2q}s_{2q+1}(\gamma) - 1) \\
> \beta_{-1/4} (n^{2p}s_{2p+1}(\gamma) - 1) - (n^{2q}s_{2q+1}(\gamma) - 1),
\end{equation}

since, by Lemma 3.2, $n^{2p}s_{2p+1}(\gamma) \geq 1$.

We prove that 
\begin{equation}
\frac{(n/2)^q - 1}{(n/2)^p - 1} \geq \frac{n^q s_q + 1(\gamma) - 1}{n^p s_p + 1(\gamma) - 1},
\end{equation}

for any $p > q$. From (5.10), with $p \rightarrow 2p$ and $q \rightarrow 2q$, we obtain $R(\gamma) > 0$ for $\gamma \geq -1/4$.

Denote $m = n/2 \geq 1$ and $u_j = 2^j s_{j+1}$ for $j \geq 0$ to rewrite (5.11) as
\begin{equation}
u_q \leq u_p (1 - \omega(m)) + \omega(m), \quad \text{where} \quad \omega(m) = \frac{mp - m^q}{m^{p+q} - m^q}.
\end{equation}

By Lemma 5.2, the sequence $\omega(m) \in (0, 1)$, is nonincreasing for $m \geq 1$. Also, by Lemma 3.2, $u_p = 2^p s_{p+1} \geq 1$, and (5.12) follows from $u_q \leq u_p [1 - \omega(1)] + \omega(1)$, where $\omega(1) = (p - q)/p$. Equation (5.12) reduces to $(u_q - 1)/q \leq (u_p - 1)/p$, for $p > q \geq 1$. In terms of $s_q(\gamma)$ this is equivalent to 
\begin{equation}
\frac{2^q s_q + 1(\gamma) - 1}{q} \leq \frac{2^p s_p + 1(\gamma) - 1}{p}, \quad \text{for} \quad p > q \geq 1,
\end{equation}

which follows for $\gamma \geq -1/4$ from Lemma 3.6, since monotonicity of the positive sequence $\frac{2^m s_{m+1} - 1}{m(m + 1)}$ directly implies monotonicity of the sequence $\frac{2^m s_{m+1} - 1}{m}$.

Thus $R(\gamma) > 0$ for all $\gamma \geq -1/4$ and $R(\gamma)$ has no roots in $[-1/4, \infty)$ and there are no collisions of eigenvalues in this regime. 

For $n = 1$, we use a similar argument. For $n = 1$ and $\gamma = 0$, $R(0) = 0$. Hence $\gamma = 0$ is always a root of $R(\gamma) = 0$, corresponding to the relation $\Omega(1) = 0 = \Omega(0)$.

For $p > q > 0$, denote
\begin{equation}
\beta_{0}^{(n-1)} = \frac{2q + 1}{2p + 1}, \quad \text{and} \quad \beta_{-1/4}^{(n-1)} = \frac{1 - 2^{-2q}}{1 - 2^{-2p}}.
\end{equation}

**Theorem 5.4. Case $n = 1$.** Let $p, q$ be positive integers with $p > q$. For the linearized problem (5.2) at zero amplitude with $c = c_1$, the presence and the character of eigenvalue collisions depend on the parameter $n$ of the Fourier mode of the perturbation as follows:

(i) for $\beta < \beta_{0}^{(n-1)}$, eigenvalues of the same signature collide, i.e., there is a root of (5.4) with $\gamma > 0$ and there is no root with $\gamma \in [-1/4, 0)$;

(ii) for $\beta > \beta_{0}^{(n-1)}$, eigenvalues of opposite signature collide, i.e., there is a root $\gamma$ of (5.4) so that $\gamma \in [-1/4, 0)$;

(iii) for $\beta_{-1/4}^{(n-1)} < \beta$, eigenvalues do not collide, i.e., $\gamma < -1/4$, for all roots $\gamma$ of (5.4).

1These eigenvalues are present due to symmetries; they do not leave the imaginary axis.
Proof. First, we show that \( \beta_{0}^{(n=1)} < \beta_{-1/4}^{(n=1)} \), which follows from the function 
\[ f(y) = (1 - 2^{-y})(1 + y) \] 
being decreasing for \( y > 2 \). Its derivative has the numerator
\[ (1 + y)2^{-y} \ln 2 + 2^{-y} - 1, \]
which is negative at \( y = 2 \), and itself has a derivative that
is negative for \( y > 2 \).
Next, for \( \beta < \beta_{-1/4}^{(n=1)} \),
\[ R(-1/4) = \beta(s_{2p+1}(-1/4) - 1) - (s_{2q+1}(-1/4) - 1) = \beta(2^{-2p} - 1) - (2^{-2q} - 1) \]
(5.13) 
where equality holds only for \( \beta = \beta_{-1/4}^{(n=1)} \). On the other hand, if \( \beta > \beta_{-1/4}^{(n=1)} \) then
\[ R(-1/4) < 0. \]
Further, for \( \gamma = 0 \) and all values of \( \beta \), \( R(0) = 0 \). Finally, for \( \gamma \in [-1/4, 0) \)
\[ R'(0) = \beta(2p + 1) - (2q + 1). \]
Therefore, for \( \beta < \beta_{0}^{(n=1)} \),
(5.14) 
\[ R(0) = 0, \quad R'(0) < 0, \]
and, for \( \beta > \beta_{0}^{(n=1)} \),
\[ R(0) = 0, \quad R'(0) > 0. \]
Note that \( R(0) = R'(0) = 0 \) for \( \beta = \beta_{0}^{(n=1)} \).

Part (i). By (5.13) one has \( R(-1/4) > 0 \), and by (5.14) \( R(0) = 0 \) and \( R'(0) < 0 \).
We prove that \( R(\gamma) > 0 \) for all \( \gamma \in [-1/4, 0) \). Thus \( R = R(\gamma) \) does not have any roots
in \(( -1/4, 0) \). Moreover, \( R(\gamma) \) is an odd-degree polynomial with a positive leading
coefficient, \( R(\gamma) \rightarrow \infty \) as \( \gamma \rightarrow \infty \) and \( R(0) = 0 \) and \( R'(0) < 0 \). Therefore \( R \) has a
positive root.

Assume \( \gamma \in [-1/4, 0) \) and \( \beta < \beta_{0}^{(n=1)} \). Then, using (3.6),
\[ R(\gamma) = \beta(s_{2p+1}(\gamma) - 1) - (s_{2q+1}(\gamma) - 1) > \beta_{0}^{(n=1)}(s_{2p+1}(\gamma) - 1) - (s_{2q+1}(\gamma) - 1). \]
To establish \( R(\gamma) > 0 \) it is enough to prove
(5.15) 
\[ \beta_{0}^{(n=1)} \leq \frac{s_{2q+1}(\gamma) - 1}{s_{2p+1}(\gamma) - 1}, \] 
for \( \gamma \in [-1/4, 0) \).
By Lemma 3.2 one has \( s_{m}(\gamma) < 1 \) for \( m \geq 2, \gamma \in [-1/4, 0) \). Hence (5.15) can be rewritten as
\[ \frac{s_{2p+1}(\gamma) - 1}{2p + 1} \geq \frac{s_{2q+1}(\gamma) - 1}{2q + 1}, \]
which follows for \( p > q > 0 \) and \( \gamma \in [-1/4, 0) \) from Lemma 3.7. Therefore \( R(\gamma) > 0 \) for
\( \gamma \in [-1/4, 0) \).

Part (ii). By (5.13) one has \( R(-1/4) > 0 \), and by (5.14) \( R(0) = 0 \), \( R'(0) > 0 \).
Therefore there exist a \( \gamma \in (-1/4, 0) \) such that \( R(\gamma) = 0 \).

Part (iii). In this case \( R(-1/4) < 0 \), and by (5.14) \( R(0) = 0 \) and \( R'(0) > 0 \). We
prove that \( R(\gamma) < 0 \) for \( \gamma \in [-1/4, 0) \) and \( R(\gamma) > 0 \) for \( \gamma > 0 \). Therefore \( R(\gamma) \) does
not have a non-zero root for \( \gamma \geq -1/4 \).
First assume that $\gamma \in [-1/4, 0)$. Then $\beta > \beta^{(n=1)}_{-1/4}$ implies, using (3.6),

$$R(\gamma) = \beta(s_{2p+1}(\gamma) - 1) - (s_{2q+1}(\gamma) - 1) < \beta^{(n=1)}_{-1/4}(s_{2p+1}(\gamma) - 1) - (s_{2q+1}(\gamma) - 1).$$

It suffices to prove

$$\beta^{(n=1)}_{-1/4} \geq \frac{s_{2p+1}(\gamma) - 1}{s_{2p+1}(\gamma) - 1}, \quad \text{for } \gamma \in [-1/4, 0),$$

to establish $R(\gamma) < 0$. The inequality (5.16) is rewritten as

$$\frac{s_{2p+1}(\gamma) - 1}{2^{-2p} - 1} \geq \frac{s_{2p+1}(\gamma) - 1}{2^{-2q} - 1},$$

which follows from Lemma 3.8. Thus $R(\gamma) < 0$ for $\gamma \in [-1/4, 0)$.

Next, we assume $\gamma > 0$. With $\beta > \beta^{(n=1)}_{-1/4}$ and using (3.7),

$$R(\gamma) = \beta(s_{2p+1}(\gamma) - 1) - (s_{2q+1}(\gamma) - 1) > \beta^{(n=1)}_{-1/4}(s_{2p+1}(\gamma) - 1) - (s_{2q+1}(\gamma) - 1).$$

It suffices to prove

$$\frac{s_{2p+1}(\gamma) - 1}{2^{-2p} - 1} < \frac{s_{2p+1}(\gamma) - 1}{2^{-2q} - 1},$$

which follows from Lemma 3.8. Thus $R(\gamma) > 0$ for $\gamma > 0$. \hfill \Box

It is easy to see that for $n = 2$, $R(-1/4) = 0$. Thus $\gamma = -1/4$ is a root of $R(\gamma) = 0$ for all $\beta$. It corresponds to the fact that $\Omega(-1) = 0 = \Omega(1)$, i.e., there is a collision of two eigenvalues of opposite Krein signature at the origin for all $\beta$. This collision is due to the symmetries of the problem and these eigenvalues do not leave the imaginary axis in the weakly nonlinear regime. Thus this collision does not affect stability. We focus on the remaining roots of $R(\gamma) = 0$.

We denote

$$\beta^{(n=2)}_0 = \frac{2q - 1}{2^{2q} - 1}, \quad \text{and} \quad \beta^{(n=2)}_{-1/4} = \frac{(2q + 1)q}{(2p + 1)2p}.$$

The inequality $\beta^{(n=2)}_0 < \beta^{(n=2)}_{-1/4}$ follows similarly to $\beta^{(n=1)}_0 < \beta^{(n=1)}_{-1/4}$, in the proof of the previous theorem.

**Theorem 5.5.** **Case n = 2.** Let $p, q, p > q$, be positive integers. For the linearized problem (5.2) at zero amplitude at $c = c_1$ the presence and the character of collisions of eigenvalues depends on the Fourier-mode parameter $n$ of the perturbation in the following way:

(i) for $\beta < \beta^{(n=2)}_0$, eigenvalues of the same signature collide, i.e. there is a root of (5.4) with $\gamma > 0$ and there is no root with $\gamma \in (-1/4, 0)$;

(ii) for $\beta^{(n=2)}_0 < \beta < \beta^{(n=2)}_{-1/4}$, eigenvalues of the opposite signature collide, i.e. there is a root $\gamma$ of (5.4) such that $\gamma \in (-1/4, 0)$;

(iii) for $\beta^{(n=2)}_{-1/4} < \beta$, eigenvalues do not collide, i.e. all roots $\gamma$ of (5.4) satisfy $\gamma \leq -1/4$.

**Proof.** **Part (i).** We prove that $R(\gamma) < 0$, for $\gamma \in (-1/4, 0)$. First, $R(\gamma)$ is an odd-degree polynomial and $R(\gamma) \to \infty$ as $\gamma \to \infty$ and $R(0) = 0$ and $R'(0) < 0$. Thus $R$ has a root $\gamma > 0$.
Assume $\gamma \in [-1/4, 0)$ and $\beta < \beta_0^{(n=2)}$. Then
\[ R(\gamma) = \beta(2^{p} s_{2p+1}(\gamma) - 1) - (2^{q} s_{2q+1}(\gamma) - 1) < \beta_0^{(n=2)}(2^{p} s_{2p+1}(\gamma) - 1) - (2^{q} s_{2q+1}(\gamma) - 1). \]

To establish $R(\gamma) < 0$ it suffices to prove
\[ \beta_0^{(n=2)} \leq \frac{2^{q} s_{2q+1}(\gamma) - 1}{2^{2p} s_{2p+1}(\gamma) - 1}, \quad \text{for } \gamma \in (-1/4, 0]. \]

This inequality is rewritten as
\[ \frac{2^{2p} s_{2p+1}(\gamma) - 1}{2^{2p} - 1} \leq \frac{2^{q} s_{2q+1}(\gamma) - 1}{2^{q} - 1}, \]

which follows from Lemma 3.5. Therefore $R(\gamma) < 0$ for $\gamma \in (-1/4, 0]$.

**Part (ii).** First,
\[ R(0) = \beta(2^{p} s_{2p+1}(0) - 1) - (2^{q} s_{2q+1}(0) - 1) = \beta(2^{p} - 1) - (2^{q} - 1) > \beta_0^{(n=2)}(2^{p} - 1) - (2^{q} - 1) = 0. \]

Next we show that $\lim_{\gamma \to -1/4^+} R'(\gamma) < 0$. Indeed, for $\gamma > -1/4$, we have
\[ R'(\gamma) = \beta \frac{2p + 1}{\sqrt{1 + 4^{p}}} 2^p (\psi^p_+ - \psi^p_-) - 2^2 (\psi^{2q}_+ - \psi^{2q}_-) < \frac{2^p + 1}{\sqrt{1 + 4^{p}}} 2^p (\psi^p_+ - \psi^p_-) - \frac{2q + 1}{\sqrt{1 + 4^{q}}} 2^q (\psi^{2q}_+ - \psi^{2q}_-), \]
as $\psi^2_+ > \psi^2_- \geq 0$. The result follows from l'Hopital’s rule, since
\[ \lim_{\gamma \to -1/4^+} \frac{(2q + 1)2^q (\psi^{2q}_+(\gamma) - \psi^{2q}_-(\gamma))}{(2p + 1)2^p (\psi^p_+(\gamma) - \psi^p_-)} = \lim_{\gamma \to -1/4^+} \frac{2q(2q + 1)2^q}{2p(2p + 1)2^p} \frac{\psi^{2q-1}_+(\gamma) + \psi^{2q-1}_-(\gamma)}{\psi^{2p-1}_+(\gamma) + \psi^{2p-1}_-(\gamma)} = \frac{2q(2q + 1)2^q}{2p(2p + 1)2^p} \frac{\psi^{2q}_+(\gamma) - \psi^{2q}_-(\gamma)}{\psi^{2p}_+(\gamma) - \psi^{2p}_-(\gamma)} = \frac{2q(2q + 1)2^q(2q-2)}{2p(2p + 1)2^p(2p-2)} = \frac{2q(2q + 1)}{2p(2p + 1)} = \beta_0^{(n=2)} - 1/4. \]

Thus $R(\gamma) < 0$ for $\gamma \in (-1/4, -1/4 + \varepsilon)$, $\varepsilon > 0$, small. Since $R(0) > 0$, there exists $\gamma \in (-1/4, 0)$ so that $R(\gamma) = 0$.

**Part (iii).** We show that $R(\gamma) < 0$ for $\gamma > -1/4$. One has
\[ R(\gamma) = \beta(2^{p} s_{2p+1}(\gamma) - 1) - (2^{q} s_{2q+1}(\gamma) - 1) > \beta_0^{(n=2)}(2^{p} s_{2p+1}(\gamma) - 1) - (2^{q} s_{2q+1}(\gamma) - 1). \]

We show that
\[ \beta_0^{(n=2)} = \frac{2q(2q + 1)}{2p(2p + 1)} \geq \frac{2^{2q} s_{2q+1}(\gamma) - 1}{2^{2p} s_{2p+1}(\gamma) - 1}, \]
which is equivalent to
\[ \frac{2^{2p} s_{2p+1}(\gamma) - 1}{2p(2p + 1)} \geq \frac{2^{q} s_{2q+1}(\gamma) - 1}{2q(2q + 1)}. \]

This inequality follows from Lemma 3.6. Therefore $R(\gamma) = 0$ has no roots $\gamma > -1/4$ for $\beta > \beta_0^{(n=2)}$. 

\[ \square \]
Remark. Note that we have assumed $\alpha = 1$ in (5.3). Thus Theorems 5.3–5.5 justify that for $\beta < 0$ there is always a collision of eigenvalues of the same signature, i.e. there is a positive root of (5.4) and no root $\gamma \in (-1/4, 0)$. That motivates our choice of signs in (5.3) and it is in an agreement with the fact that for $\alpha > 0$ and $\beta < 0$ the Hamiltonian (2.2) associated with (5.1) is definite. Therefore all eigenvalues have the same Krein signature and no Hamiltonian-Hopf bifurcations are possible. Hence the case $\alpha \beta > 0$ represents the interesting case in which the associated Hamiltonian is indefinite and contains balancing terms.

REFERENCES