Center for Mathematics and Computer Science
rademach@cwi.nl - www.cwi.nl/∼rademach

Computing essential and absolute spectra by continuation

Stability Workshop, University of Washington, Sept. 2006

Jens Rademacher (CWI Amsterdam)

Abstract eigenvalue problem

We consider eigenvalue problems for operators on the real line that can be cast as linear (non-autonomous) ODE

$$v_x = A(x,\lambda)v\ ,\quad x\in\mathbb{R}, \lambda\in\mathbb{C}, v\in\mathbb{R}^n$$

where $A(x,\lambda)$ is analytic in λ.

These arise naturally for steady solutions of parabolic PDEs, e.g. KdV, (coupled) NLS, CGL, reaction-diffusion equations, ...

Usually: bounded solution $\Rightarrow \lambda$ in spectrum.
Localized nontrivial solution $\Rightarrow \lambda$ eigenvalue in point spectrum.

Asymptotically constant / periodic $A(\cdot,\lambda)$:

essential spectrum spectrum bounded by, absolute spectrum given by spectra of asymptotic states.

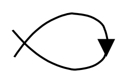
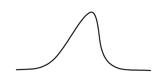
Spatial Dynamics and Spectral ODE

Prototype: **Reaction diffusion system (RDS)**

$$U \in \mathbb{R}^N,\; x \in \mathbb{R}, \quad U_t \quad = \quad DU_{xx} + cU_x + F(U)$$

Existence of t.w.: **Equilibria satisfy travelling wave ODE**

$$0 = DU_{xx} + cU_x + F(U) \quad \Leftrightarrow \quad u_x = f(u;c) \quad u \in \mathbb{R}^{N+\dim(\mathrm{Rg}(D))}$$

homoclinic  $\leftrightarrow$ **pulse** 

periodic orbit $\leftrightarrow$ **wave train, heteroclinic** $\leftrightarrow$ **front**

Stability: **Eigenvalue problem of linearization in travelling wave**

$$\lambda V = \mathcal{L}V := DV_{xx} + cV_x + \partial_U F(U(x))V \;\Leftrightarrow\; v_x = a(x)v + \lambda Bv$$

Assume $U(x)$ **constant or periodic,** $A(x,\lambda) = a(x) + \lambda B$**.**

Spatial eigenvalues and dispersion

Eigenvalue problem is linear (non-autonomous) ODE

$$v_x = A(x,\lambda)v$$

Complex dispersion relations ($\lambda, \nu \in \mathbb{C}$):

$$A(x,\lambda) \equiv A(\lambda): \quad d(\lambda,\nu) \quad := \quad \det(A(\lambda) - \nu) = 0$$

$$A(x,\lambda) = A(x+L,\lambda): \quad d(\lambda,\nu) \quad := \quad \det(\Phi(L;\lambda) - \mathrm{e}^{\nu L}) = 0$$

Here $\Phi(L;\lambda)$ is the period map of the evolution of $v_x = A(x,\lambda)v$.

Call such ν spatial eigenvalues or spatial Floquet exponents.

Simplest example:

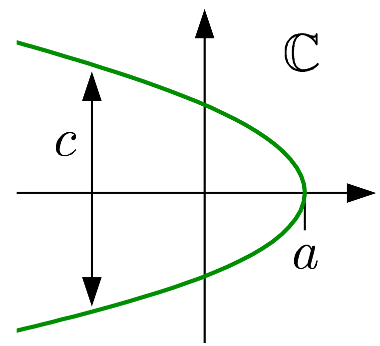
$$u_{xx} + cu_x + au = \lambda u$$

$$d(\lambda,\nu) = \nu^2 + c\nu + a - \lambda$$

Essential spectrum on $\mathbb{R}$: λ vs. $\nu = \mathrm{i}\gamma$

Essential spectrum: $\lambda \in \Sigma_{\mathrm{ess}} \;\Leftrightarrow\; \exists \gamma \in \mathbb{R} : d(\lambda, \mathrm{i}\gamma) = 0$.

Simplest example:



$$d(\lambda, \mathrm{i}\gamma) = -\gamma^2 + c\mathrm{i}\gamma + a - \lambda$$

$$\Sigma_{\mathrm{ess}} = \{-\gamma^2 + c\mathrm{i}\gamma + a\}$$

(as from Fourier transform).

Generally: **Essential spectrum is given by two real equations, three real unknowns $\Rightarrow$ curves by implicit function theorem whenever $\partial_\lambda d(\lambda(\gamma), \mathrm{i}\gamma) \neq 0$.**

Continuation

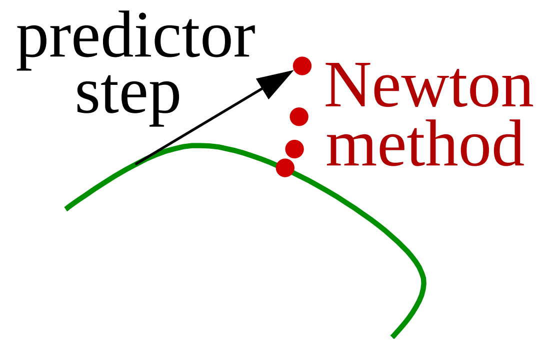
Continuation numerics:

Newton method,

arclength parametrization,

parameter switching

[e.g. Allgower, Georg].



Bad: **Need initial conditions. Computes spectrum locally (can continue several curves simultaneously).**

Nice: **Versatile, robust, very accurate. Can pathfollow spectrum in parameters of nonlinear problem $\rightarrow$ locate and determine type of onset of instability etc.**

Σ_{ess} for constant coefficients

Always connected set in $\bar{\mathbb{C}}$.

RDS: $\Sigma_{\text{ess}} = \cup_{j=1}^{N} \{\lambda_j(\gamma) : \gamma \in \mathbb{R}\}$, $\text{Re}(\lambda) \to -\infty \Leftrightarrow \gamma \to \infty$, **stability independent of c. A priori bound for critical spectrum:**
$d(\mathrm{i}\omega, \mathrm{i}\gamma) = 0 \Leftrightarrow |\omega| \leq |c|R_0$ **and** $|\gamma| \leq R_0$, **where for** $b = \partial_U F(U)$
$R_0 = \max_{j=1,\ldots,N}(|b_{jj}| + \sum_{i=1,i\neq j}^{N} |b_{ij}|)/d_j$.

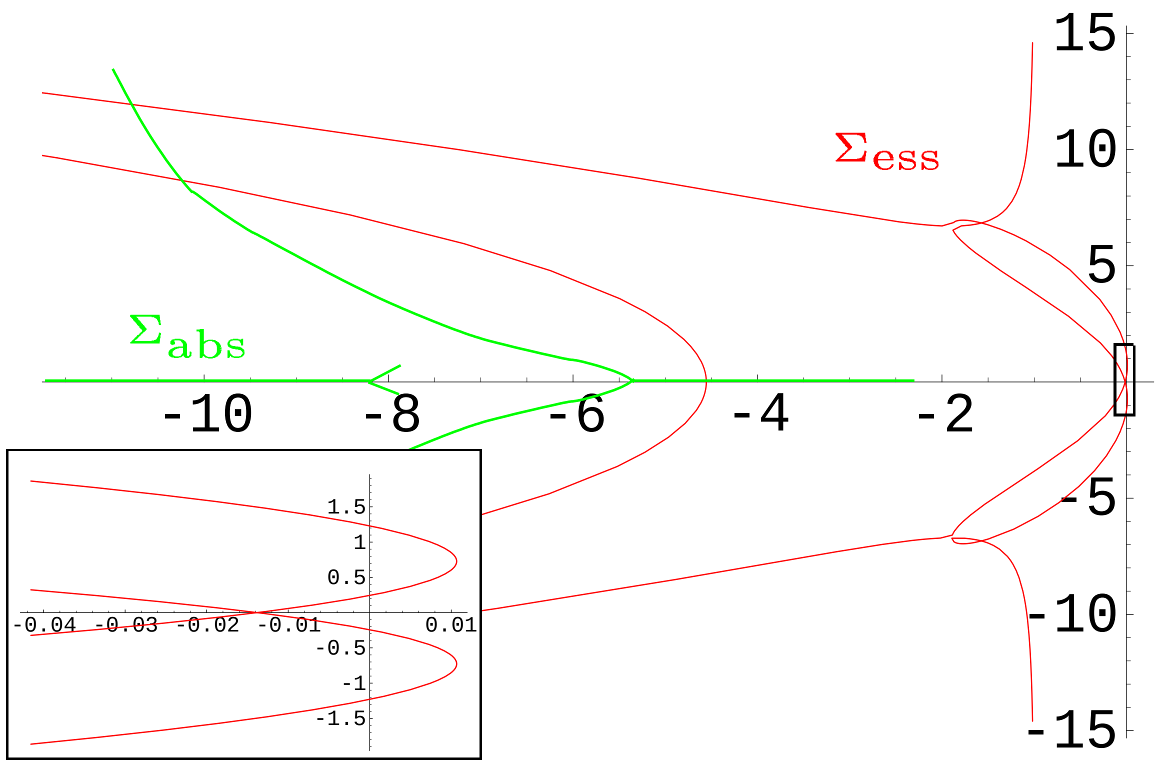
Numerics: **Dispersion relation as matrix eigenvalue problem**
in $\mathbb{R}^n$: $[A(\lambda) - \mathrm{i}\gamma]u = 0$. **RDS in $\mathbb{R}^N$:** $[-D\gamma^2 + c\mathrm{i}\gamma + b - \lambda]u = 0$.

Symmetry: **Normalize eigenvector by** $\langle \partial_\gamma u, u \rangle = 0$.
For numerics: $\langle u_{\text{old}}, u \rangle = 1$ **with** u_{old} **from previous step.**

Initial points for RDS: λ **eigenvalue of linearized kinetics at** $\gamma = 0$.

Example for constant coefficients in the Oregonator:

$$
\begin{aligned}
u_t &= D_u u_{xx} + c u_x + (u(1-u) - v(u-q))/\epsilon \\
v_t &= D_v v_{xx} + c v_x + (fw + \phi - v(u+q))/\delta \\
w_t &= c w_x + u - w
\end{aligned}
$$



Σ_{ess} for periodic coefficients

Countably many, bounded curves (Bloch decomposition):

$$\Sigma_{\mathrm{ess}} = \cup_{j=1}^{\infty} \{\lambda_j(\gamma) : \gamma \in [0, 2\pi/L)\}.$$

May contain isolated closed curves.

Recall dispersion relation: $d(\lambda, \nu) = \det(\Phi(L; \lambda) - \mathrm{e}^{\nu L}) = 0$

As BVP: $v_x = A(x, \lambda)v\ ,\quad v(L) = v(0)\mathrm{e}^{\mathrm{i}\gamma L}$**,** $x \in [0, L]$**.**

For RDS numerics solve linear and nonlinear in tandem:

$$\begin{aligned} u_x &= Lf(u; c) & u(1) &= u(0) \\ v_x &= L[f'(u(x); c) + \lambda B - \mathrm{i}\gamma]v & v(1) &= v(0) \end{aligned}$$

fix phase: $\int_0^1 \langle u_x, u_{\mathrm{old}} - u\rangle = 0$**, fix eigenfunction:** $\int_0^1 \langle v_{\mathrm{old}}, v\rangle = 1$**.**

Initial conditions e.g. from periodic case $\gamma = 0$ **and discretization (domain** $[0, 1]$**!). For RDS** u_x **is eigenfunction for** $\lambda = \gamma = 0$**.**

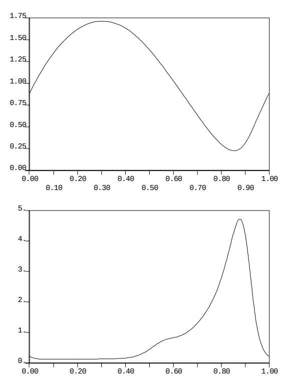
Example for periodic coefficients

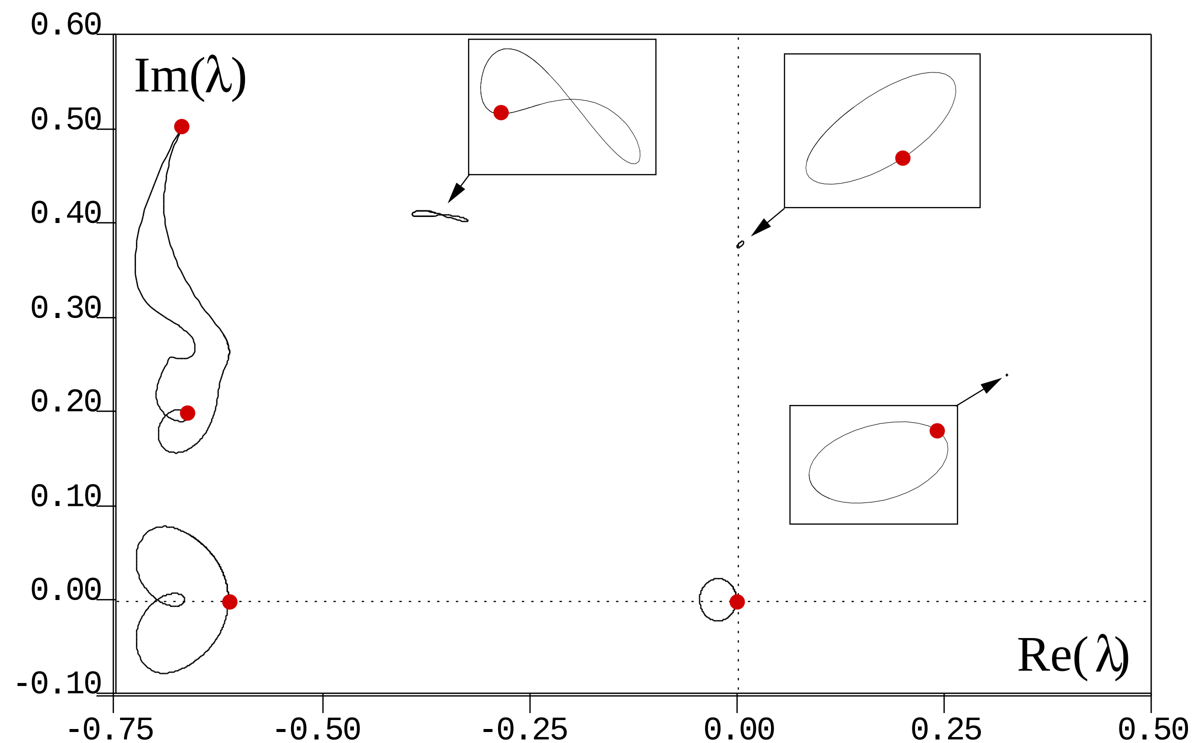
A wave train in the Schnakenberg model ($d \approx 0.45$):

$$\begin{aligned} u_t &= du_{xx} + 0.029u_x + 0.1 - uv^2 \\ v_t &= 0.01dv_{xx} + 0.029v_x + 0.9 - v + uv^2 \end{aligned}$$

Essential spectrum in region about origin:

Profile:





Testing stability

Constant:

Since connected, continue $d(\mathrm{i}\omega, \nu) = 0$ for all ν in $\omega \in [0, R_0]$, find initial points as matrix eigenvalues. Stable $\Leftrightarrow \mathrm{Re}(\nu) \neq 0$.

Periodic:

1. **Stable near $\lambda = 0$, i.e. curve at zero has tangency into $\mathrm{Re} < 0$.**
2. **Stable for $\gamma = 0$, i.e. on periodic domain $[0, L]$, find by discretizing linear operator.**
3. **Same as for constant with analogous a priori bound. Can find all ν's by Newton method.**

Note: Need not compute curves of spectrum for this.

Meaning of the absolute spectrum

On bounded domain of length L, only point spectrum:

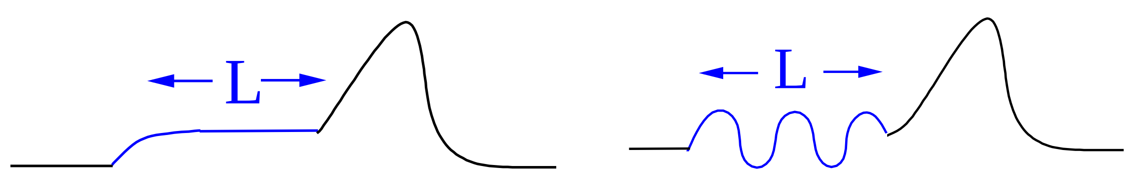
Convective vs. absolute instability:

(Assume stable point spectrum and stable 'resonance poles')

- $\Sigma_{\rm abs}$ **stable,** $\Sigma_{\rm ess}$ **unstable: perturbations are convected through the boundary.**
- $\Sigma_{\rm abs}$ **unstable: Instability, perturbation grow pointwise if point in** $\Sigma_{\rm abs}$ **with zero group velocity is unstable.**

As $L \to \infty$ point spectrum 'clusters':

- **For periodic b.c. at** $\Sigma_{\rm ess}$**, but separated b.c.: at** $\Sigma_{\rm abs}$**.**
- **On** $\mathbb{R}$ **at part of** $\Sigma_{\rm abs}$ **if profiles shadow const./per. solution:**



Lin. spreading speed: $\Sigma_{\rm abs}(c_*) \cap \mathrm{i}\mathbb{R} \neq \emptyset$ **and** $\Sigma_{\rm abs}(c_*) \cap \{\mathrm{Im} > 0\} = \emptyset$**.**

The absolute spectrum

Let Σ_L be the spectrum of the travelling wave on $(-L, L)$ with separated boundary conditions.

$$\Sigma_{\mathrm{abs}} := \{\lambda \in \mathbb{C} \text{ is an accumulation point of } \Sigma_L \text{ as } L \to \infty\}$$

Assume $\exists \rho : \mathrm{Re}(\nu) \neq 0$ **for** $\mathrm{Re}(\lambda) \geq \rho$. **Take** $d(\lambda, \nu) = 0 \to \nu(\lambda)$.

Theorem [San.Sch.] Order $\mathrm{Re}(\nu_j(\lambda)) \geq \mathrm{Re}(\nu_{j+1}(\lambda))$**, then**

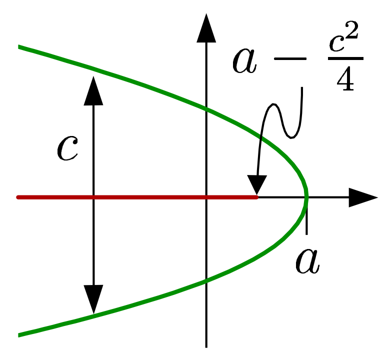
$$\Sigma_{\mathrm{abs}} = \{\mathrm{Re}(\nu_{i_\infty}(\lambda)) = \mathrm{Re}(\nu_{i_\infty+1}(\lambda))\}.$$ **RDS,** $D > 0 : i_\infty = N.$

Simplest example: $u_{xx} + cu_x + au = \lambda u$

$\nu^2 + c\nu + a = \lambda \to \quad \nu_\pm = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - a + \lambda}$

$$\Sigma_{\mathrm{abs}} = \{\lambda \leq a - \tfrac{c^2}{4}\}$$

$= \{a - \frac{c^2+\gamma^2}{4} : \gamma \geq 0\}$, $\mathrm{i}\gamma = \nu_+ - \nu_-$



Absolute spectrum by continuation

Generalized abs. spec. $\Sigma^*_{\rm abs}$**:** $\quad d(\lambda,\nu_1)=d(\lambda,\nu_2)=0$, $\nu_1-\nu_2=\mathrm{i}\gamma$.

Six real equations, seven unknowns $\to$ **curves, continue e.g. in** γ.

Write as coupled eigenvalue problems

$$\begin{aligned} u_j' &= (A(\lambda)-\nu_j)u_j, \\ \nu_1-\nu_2 &= \mathrm{i}\gamma. \end{aligned}$$

Regularize $\nu_1=\nu_2$

($u_1=u$, $u_2=u+\mathrm{i}\gamma v$**)**

$$\begin{aligned} u' &= (A(\lambda)-\nu)u, \\ v' &= (A(\lambda)-(\nu+\mathrm{i}\gamma))v-u \end{aligned}$$

Normalize: $\int_0^1\langle u_{\rm old},u\rangle=1$, $\int_0^1\langle v,u_{\rm old}\rangle+\langle u,v_{\rm old}\rangle+\mathrm{i}\gamma\langle v,v_{\rm old}\rangle=0$.

Initial points: 'branch points' $\gamma=0$, **i.e.** $d(\lambda,\nu)=\partial_\nu d(\lambda,\nu)=0$.

Continue $\mathrm{Re}(\nu_1-\nu_2)$ **to zero... not systematic for periodic case.**

Structure of absolute spectrum

Constant case:

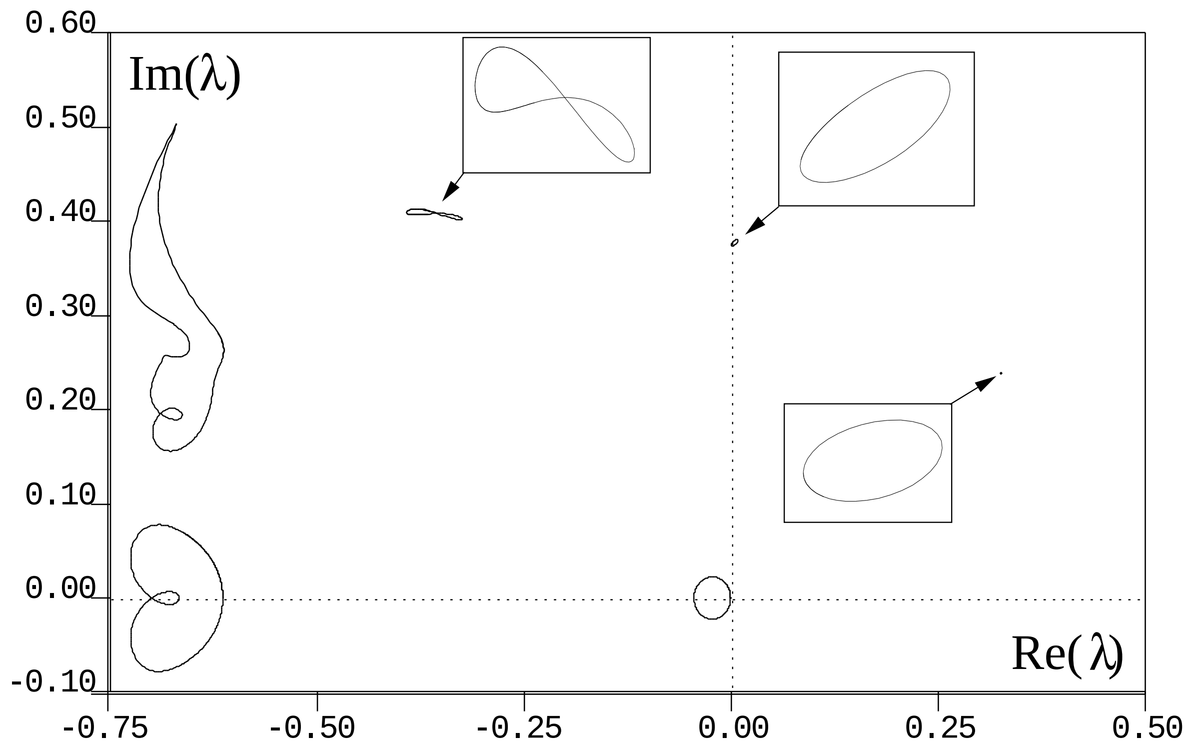
Theorem [San.Sch.R.] Σ_{abs} **is a connected set in** $\bar{\mathbb{C}}$**, i.e. stable** $\Leftrightarrow$ $\Sigma_{\text{abs}} \cap \mathrm{i}[0, R_0] = \emptyset$**. RDS:** $\Sigma^*_{\text{abs}} = \{\lambda_j(\gamma) : \gamma \geq 0, j = 1 \ldots \binom{2N}{2}\}$**, i.e. can start at branch points (compute from resultant) to get all.**

Periodic case:

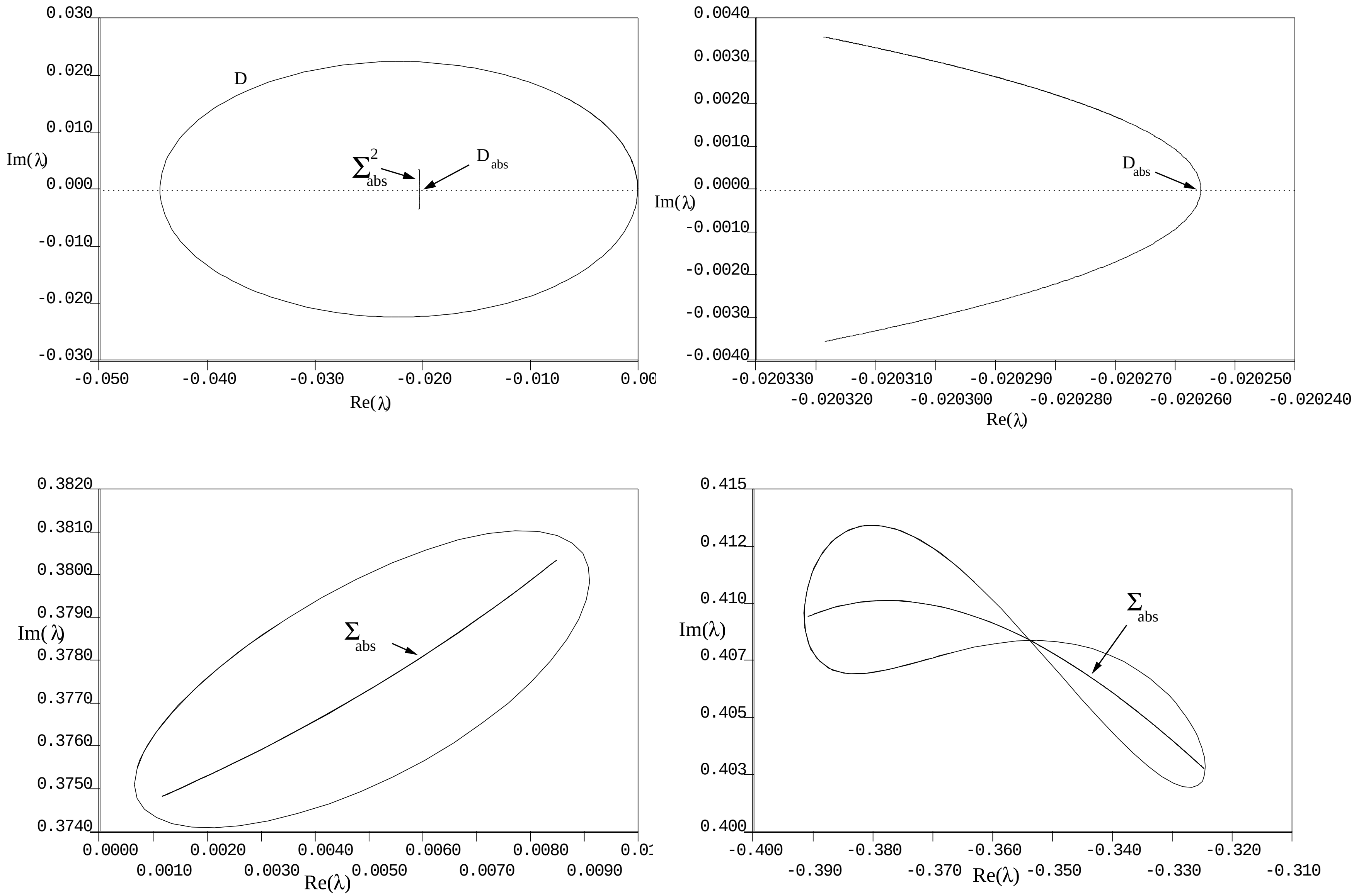
Theorem [R.] **Interior of (regular) isolated curves of** Σ_{ess} **contain** Σ^*_{abs}**. Such curves in the boundary of the most unstable connected component of** $\mathbb{C} \setminus \Sigma_{\text{ess}}$ **contain** Σ_{abs}**, i.e. then** Σ_{abs} **disconnected set.**

Schnakenberg example revisited

Essential spectrum in region about origin:



Schnakenberg example revisited



Testing stability

Constant:

Since connected, continue $d(\mathrm{i}\omega, \nu) = 0$ for all ν in $\omega \in [0, R_0]$, find these as matrix eigenvalues. Stable $\Leftrightarrow \mathrm{Re}(\nu_{i_\infty}) \neq \mathrm{Re}(\nu_{i_\infty+1})$.

Periodic:

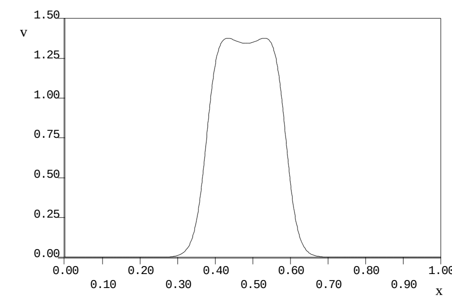
No systematic test known...

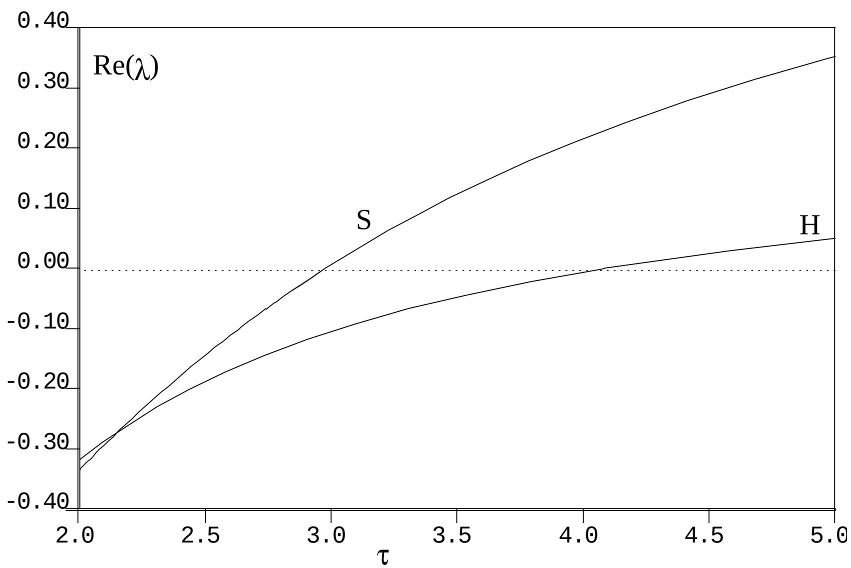
Do not know how to locate branch points...

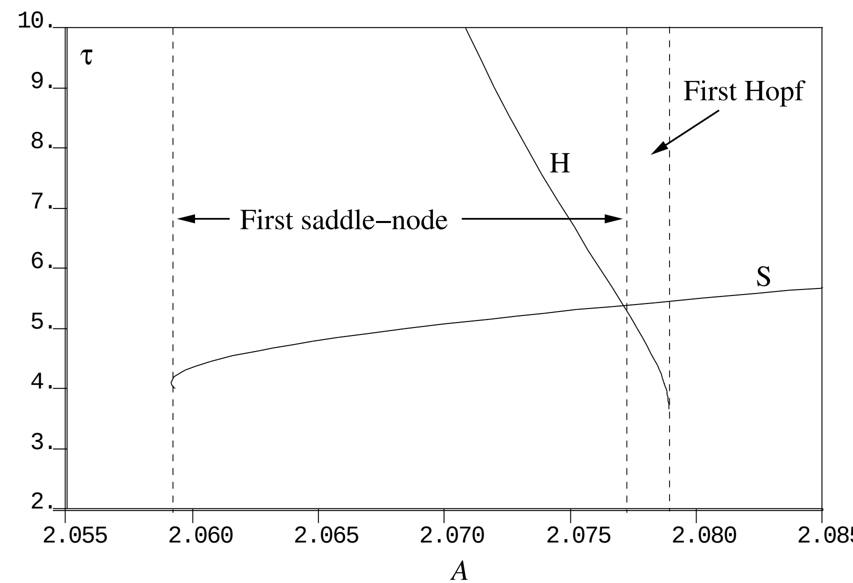
(Sufficient for instability is isola in left half plane and most unstable component of $\mathbb{C} \setminus \Sigma_{\mathrm{ess}}$.)

Instability thresholds in Gray-Scott model

$$
\begin{aligned}
v_t &= 0.001 v_{xx} - v + A u v^2 \\
\tau u_t &= 0.002 u_{xx} + 1 - u - u v^2
\end{aligned}
$$

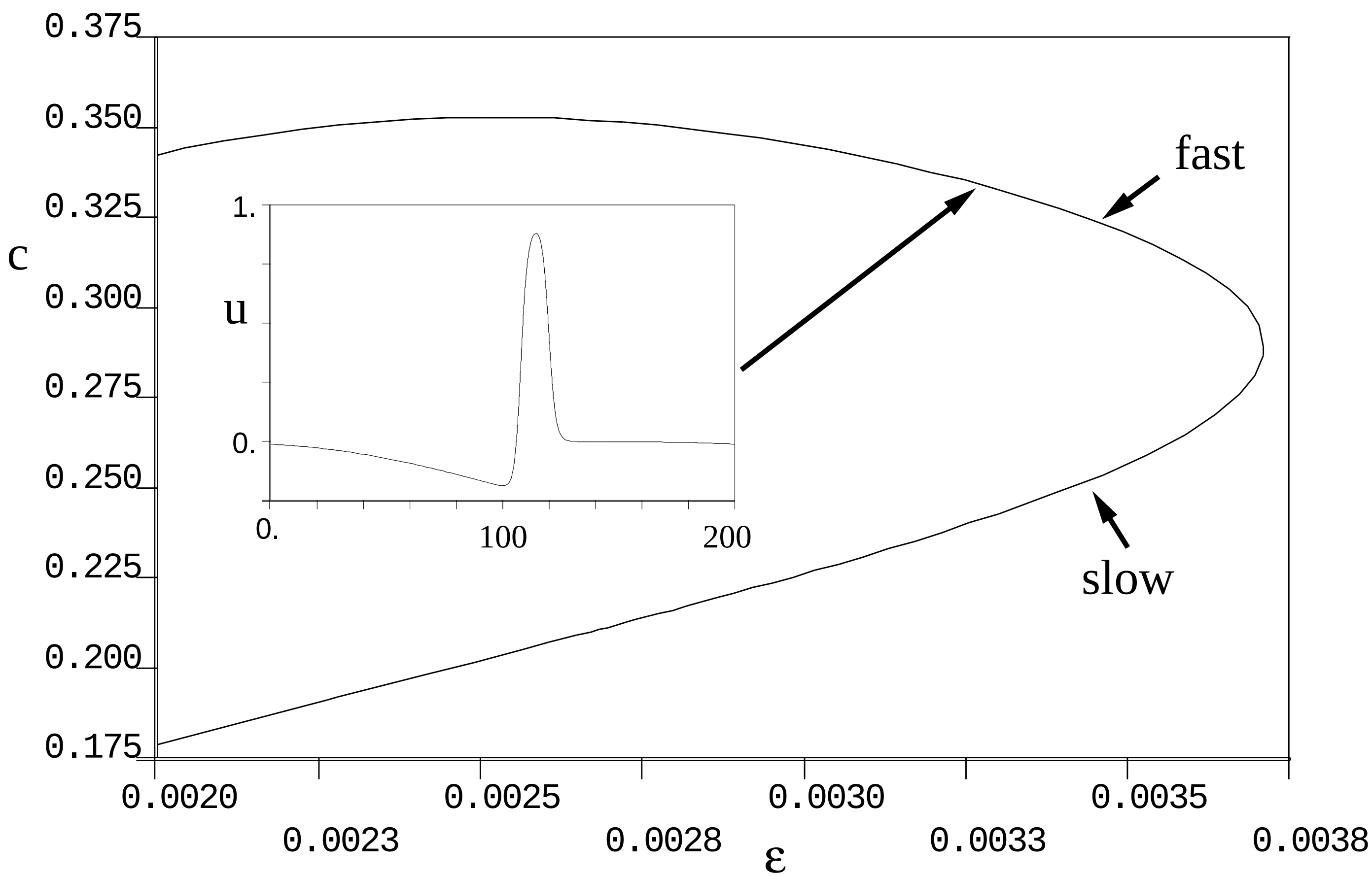






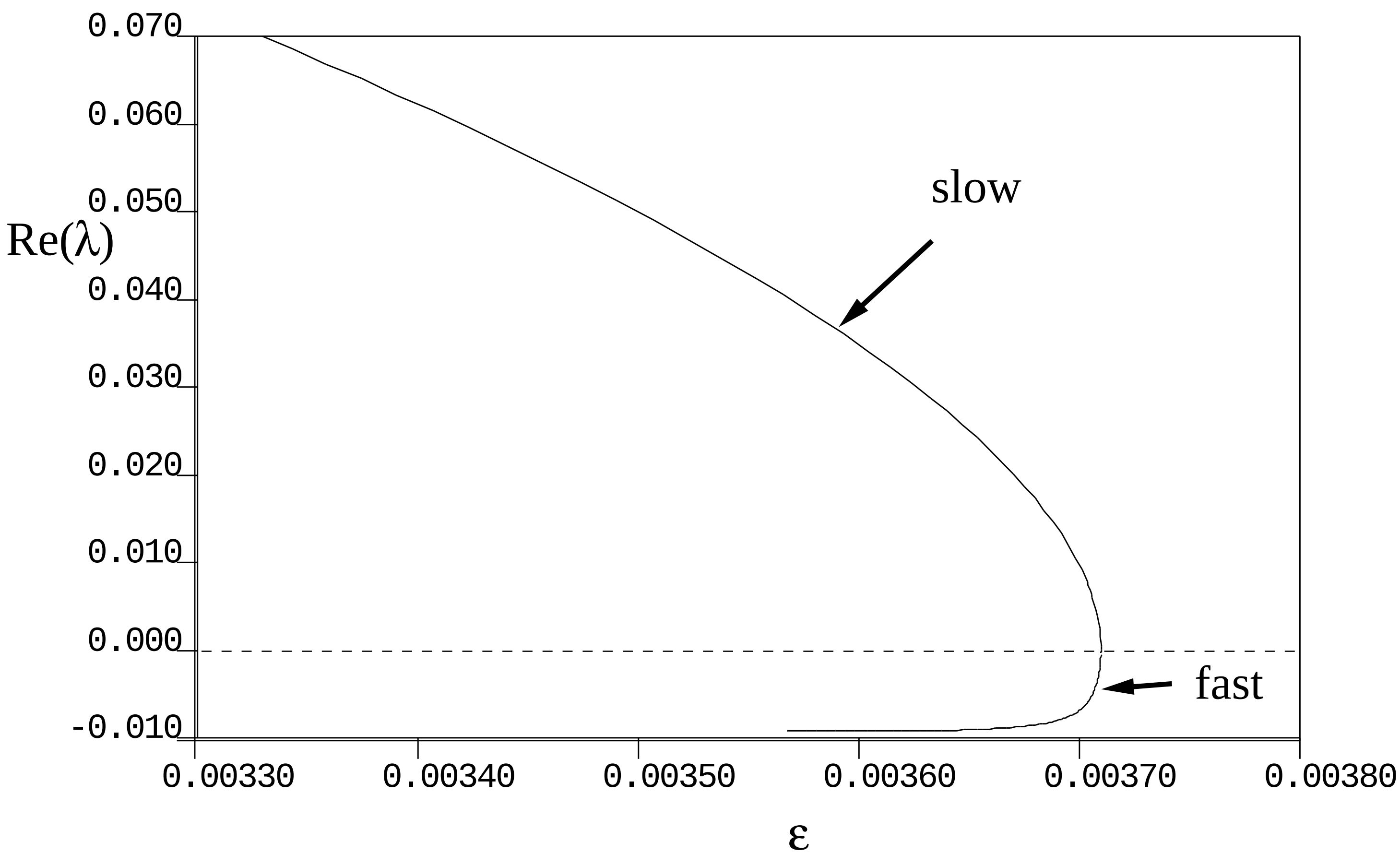
FitzHugh-Nagumo equations

$$
\begin{aligned}
u_t &= u_{xx} + cu_x - v - u(u-1)(u-a) \\
v_t &= \delta v_{xx} + cv_x + \epsilon(u - \gamma v),
\end{aligned}
$$



Fold of FHN wave train

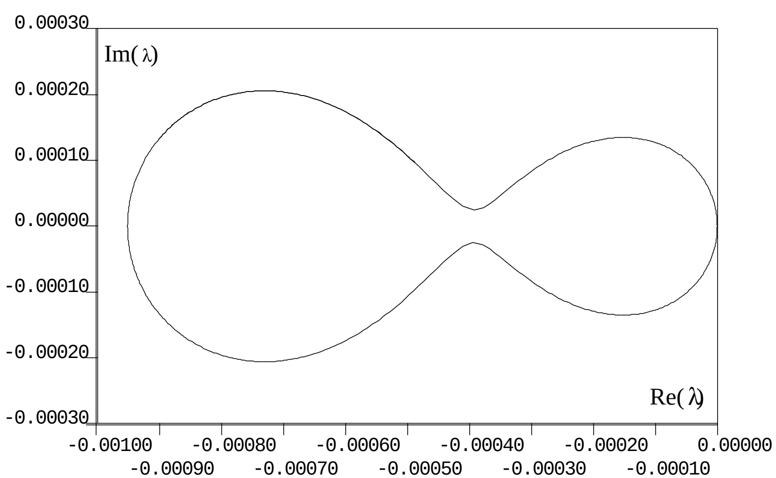
At fold point real eigenvalue for periodic domain $[0, L]$ crosses:

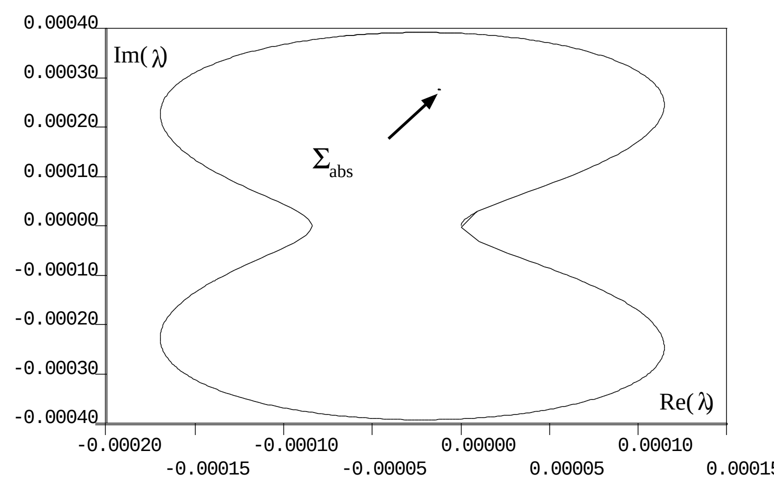


But on $\mathbb{R}$ have the whole essential spectrum!

FHN instability on $\mathbb{R}$ via isolas

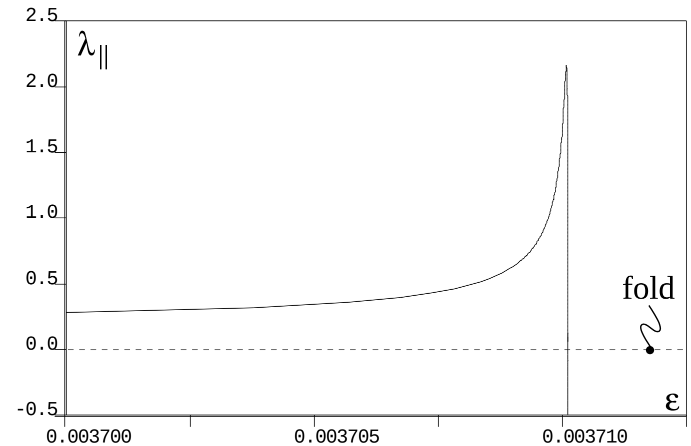
1. Two separated isola, one at origin and $\mathrm{Re}(\lambda) \leq 0$
2. Both isola merge in figure eight shape
3. Combined isola flips into unstable half plane before fold point: side-band instability.
4. At fold point: two points with vertical tangent touch at origin
5. Isola split into two, both in unstable half plane





Instability onset on $\mathbb{R}$: tangency coefficient

The tangency coefficient $\lambda_{||}$ changes sign: onset occurs at zero wave number.



Computed via $\lambda_{|} := \left.\frac{\mathrm{d}\lambda_0}{\mathrm{d}\nu}\right|_{\nu=0}, \qquad \lambda_{||} := \left.\frac{\mathrm{d}^2\lambda_0}{\mathrm{d}\nu^2}\right|_{\nu=0}$**:**

$$
\begin{aligned}
V_{|}' &= A(x,\lambda)V_{|} + [\lambda_{|}B - 1]V \\
V_{||}' &= A(x,\lambda)V_{||} + 2[\lambda_{|}B - 1]V_{|} + \lambda_{||}BV
\end{aligned}
$$

The complex Ginzburg–Landau equation

$$A_t = (1+\mathrm{i}\alpha)A_{xx} + A - (1+\mathrm{i}\beta)A|A|^2$$

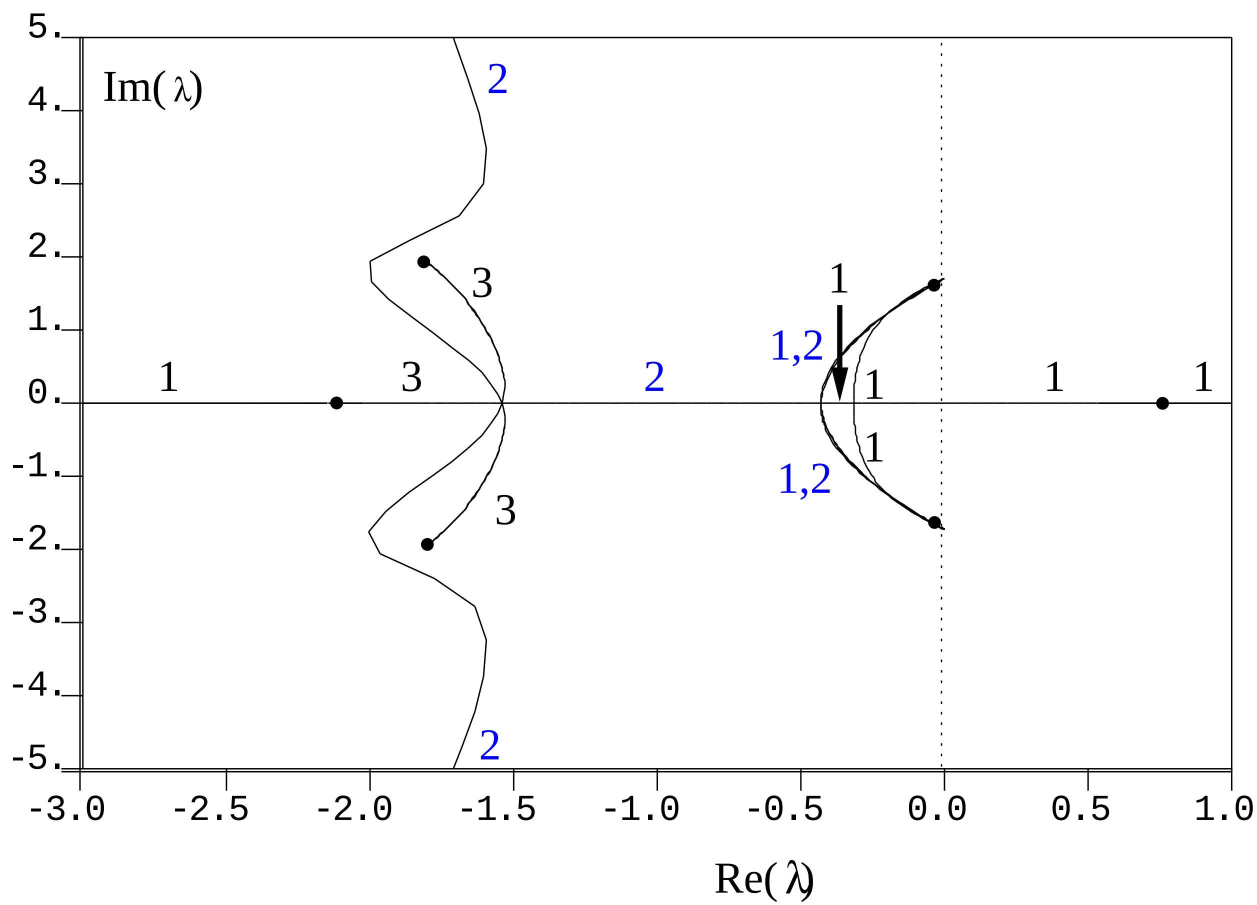
has periodic wave-trains $A_* = r\mathrm{e}^{\mathrm{i}(\kappa x - \omega t)}$ **with** $r^2 = 1-\kappa^2$ **and** $\omega = \beta + (\alpha-\beta)\kappa^2$**. In detuned variable** $A = \tilde{A}\mathrm{e}^{-\mathrm{i}\omega t}$ **CGL with c.c. like RDS for** $N=2$ **with constant coefficients:** $d(\lambda,\nu) =$

$$\begin{vmatrix} (1+\mathrm{i}\alpha)(\nu^2+2\mathrm{i}\kappa\nu) - (1+\mathrm{i}\beta)r^2 - \lambda & -(1+\mathrm{i}\beta)r^2 \\ -(1+\mathrm{i}\beta)r^2 & (1-\mathrm{i}\alpha)(\nu^2-2\mathrm{i}\kappa\nu) - (1-\mathrm{i}\beta)r^2 - \lambda \end{vmatrix}$$

Recall ordering $\mathrm{Re}(\nu_j) \geq \mathrm{Re}(\nu_{j+1})$**. Here:** $\mathrm{Re}(\nu_2) = \mathrm{Re}(\nu_3) \to \Sigma_{\mathrm{abs}}$**.**

CGL absolute spectrum

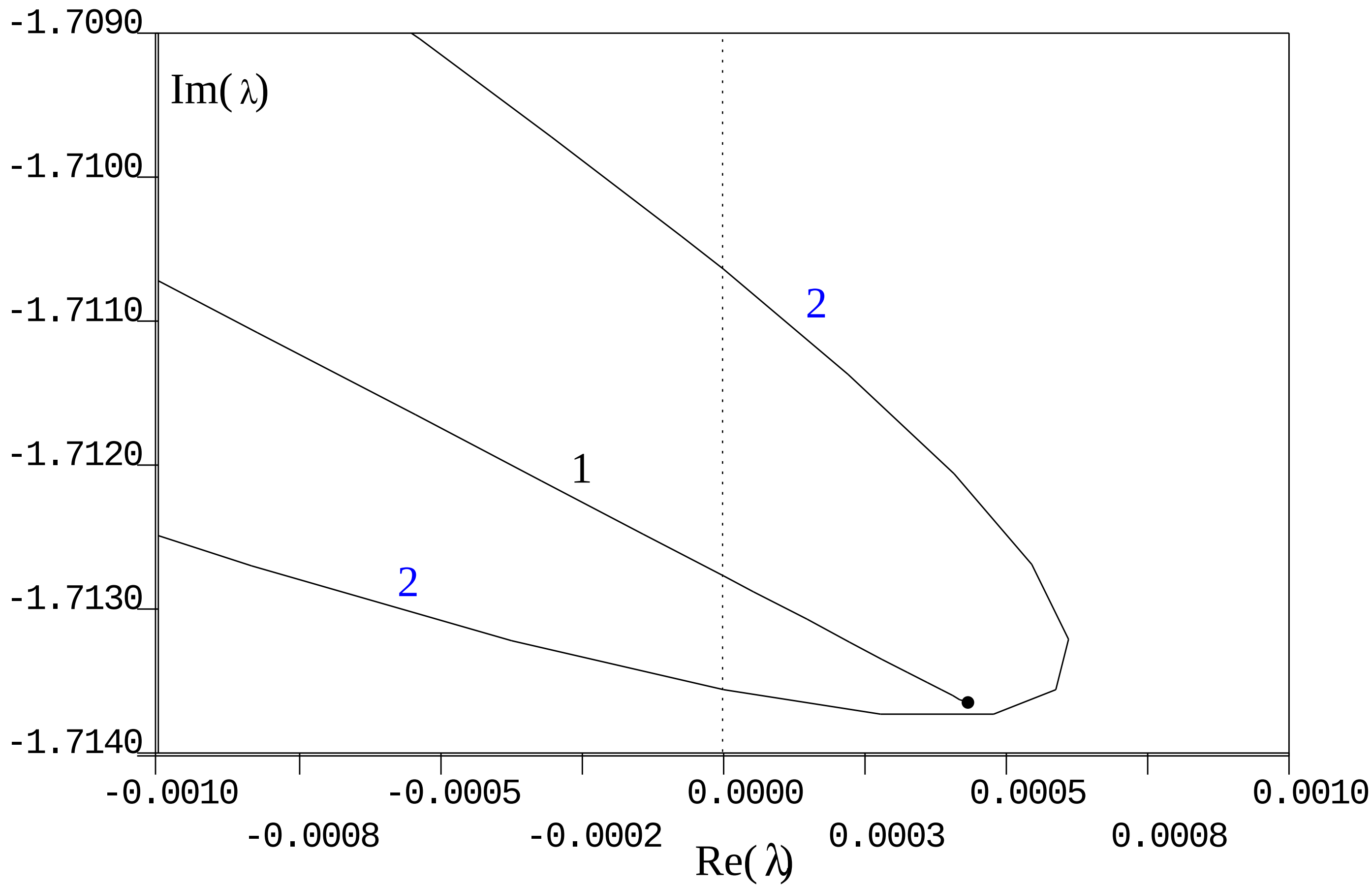
Benjamin-Feir unstable: $\alpha = -8$, $\beta = 1$, $\kappa = -0.3$



Numbers are j **where** $\mathrm{Re}(\nu_j) = \mathrm{Re}(\nu_{j+1})$. $j = 2$: Σ_{abs}

CGL absolute spectrum

Magnify one of the critical regions:



There is no branch point in the absolute spectrum $\rightarrow$

Cannot determine instability by looking at branch points alone!

References

- J.R., B. Sandstede and A. Scheel. Computing absolute and essential spectra using continuation. IMA Preprint No. 2054 (2005).
- B. Sandstede and A. Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. Physica D 145 (2000) 233-277.
- J.R. Geometric relations of absolute and essential spectra of wave trains. Accepted at SIADS (2006).
- review: B. Sandstede. Stability of travelling waves. In: Handbook of Dynamical Systems II (ed. B. Fiedler). North-Holland (2002) 983-1055.
- E. Doedel et al. *AUTO2000: Continuation and bifurcation software for ordinary differential equations (with HOMCONT)*. Technical report, Concordia University, 2002.