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Nonlinear, Anisotropic, Shell for Biologic Tissue

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Introduction: We describe the development of a unique plane-stress shell for biologic tissue. The numerical difficulties associated with incompressibility and non-linearity are addressed within the framework of an explicit integration built upon a corotational form of the well-known uniformly reduced Hughes-Liu shell [2]. The deformation gradient is derived by relating the deformation of two time-consecutive deformations that have been rectified to a quasi-material basis. Continuity requires that the spatial Cauchy stresses be remapped to a local element basis. The resultant shell can accommodate any arbitrary nonlinear anisotropic constitutive description and is objective for arbitrarily large strains that do not violate the plane-stress hypothesis.

Kinematics: In nonlinear continuum mechanics, the deformation gradient, **F**, allows the spatial position - and length - of a line element, or vector, at an arbitrary deformation to be described in terms of its material (or reference) position - and length. Inevitably, in a general three-dimensional deformation field, the deformation gradient will map rigid body translation and rotations as well as stretch. Within the context of a shell or membrane element, the advantages of adopting a corotational coordinate system to obtain **F** are twofold. First, rigid body motions are excluded. Second, the thickness stretch increment will always be referred to the same corotational normal axis. This latter consideration is especially important because it allows the thickness increment, which is 'virtual' in terms of numerical state-space, to be derived in terms of the kinematic statement of incompressibility, $det\left[?\mathbf{F}\right] = 1$. This completes the incremental deformation gradient, related to the incremental displacement gradient, D**G**, by

$$?\mathbf{F} = ?\mathbf{G} + \mathbf{I} \qquad [1]$$

The current deformation gradient $\mathbf{F}_{t+\Delta t}$ may now be related to the deformation of the preceding time-step, $\mathbf{F}_t$, as

$$\mathbf{F}_{t+\Delta t} = ?\mathbf{F} \cdot \mathbf{F}_t \qquad [2]$$

where $\mathbf{F}_{t=0}$ is simply **I**. The above relationship is exact. Accuracy is only limited by D**G** that derives immediately from the shape functions of the Hughes-Liu shell[1].

The disadvantage of a corotational system is that the local x-axis is defined by the ray connecting the first and second nodes. As the element deforms, therefore, so does the coordinate basis, even though the unit normals, by definition, remain co-linear. The deformation gradient, however, relates the deformed configuration to the undeformed configuration. The incremental deformation gradient must, therefore, be rectified to a quasi-global coordinate system before integration. Rectification flows directly from the definition of the deformation gradient. If **N** is the vector defining the material x-axis, **n**=**FN** is that vector in the spatial (deformed) configuration. Specifically, if **N** =[1 0 0], **n** is given by

$$\mathbf{n} = \begin{bmatrix} \frac{F_{11}}{\sqrt{C_{11}}} & \frac{F_{21}}{\sqrt{C_{11}}} & \frac{F_{31}}{\sqrt{C_{11}}} \end{bmatrix} \qquad [3]$$

where **C** is the right Cauchy-Green tensor, $\mathbf{C}=\mathbf{F}^{\mathbf{T}}\mathbf{F}$. The rotation is obtained as

$$?\mathbf{G}' = \mathbf{M}^{\mathbf{T}} \cdot ?\mathbf{GM} \qquad [4]$$

with **M** defined as

$$\mathbf{M} = \begin{bmatrix} n_1 & -n_2 & 0 \\ n_2 & n_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [5]$$

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[1] A complete description of the Hughes-Liu shell is beyond the scope of this paper. The interested reader is referred to [1,2];

Kinetics: We are interested in Cauchy or true stress, i.e. force per deformed area. To articulate a stress-strain mapping based on a suitable strain-energy function, it is necessary to relate two work conjugates. The second Piola-Kirchoff stress will, therefore, be mapped from the right Cauchy-Green tensor though a nonlinear strain energy function:

$$S = 2 \cdot \frac{\partial ?}{\partial \mathbf{C}} \qquad [6]$$

where, for conventional orthotropy, for example, Ψ, the strain energy function, will be in terms of invariants one to seven of **C**. Cauchy stress is obtained by "pushing forward" the second Piola-Kirchoff stress

$$\frac{1}{J}\mathbf{F} \cdot \mathbf{S}\mathbf{F}^{\mathbf{T}} + p\mathbf{I} \qquad [7]$$

Note that p is a Lagrange multiplier usually associated with hydrostatic pressure to enforce incompressibility. Here, however, since we have already enforced incompressibility while integrating the deformation gradient, p enforces the plane-stress assumption.
Finally, the Cauchy stress must be rotated back to the element coordinate system

$$\mathbf{s} = \mathbf{M} \cdot \mathbf{s}'\mathbf{M}^{\mathbf{T}} \qquad [8]$$

Validation: Figure 2 shows the results from a general finite stretch and shear of a strain energy function for porcine mitral valve [3].

$$\mathbf{s} = 2 \cdot W_I \mathbf{B} + \left(\frac{W_a}{a}\right) \cdot \mathbf{F} \cdot \mathbf{N} \otimes \mathbf{N} \cdot \mathbf{F}^{\mathbf{T}} + p\mathbf{I} \qquad [9]$$

where $\mathbf{W}_I$ is $\frac{\partial \Psi}{dI}$, $\mathbf{W}_\alpha$ is $\frac{\partial \Psi}{da}$, I is the first invariant of **C**, and α is the fiber-direction stretch in an arbitrary material direction **A** given by $a^2 = \mathbf{A} \cdot \mathbf{C}\mathbf{A}$ [10]

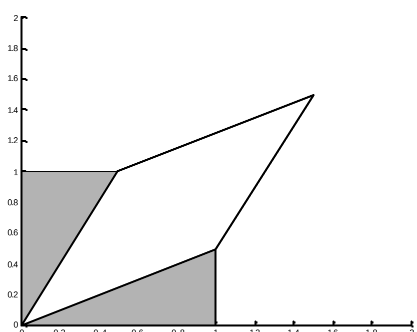

Fig.1 – Deformed (gray), deformed (white).

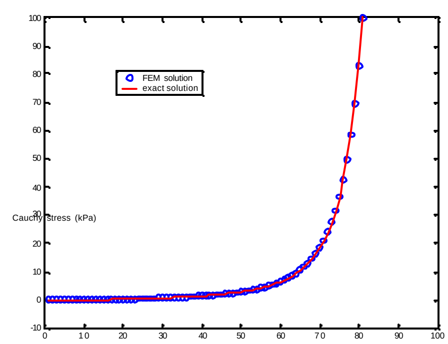

Fig. 2 – X-axis Cauchy stress: FEM vs. exact.

Conclusion: We have successfully developed a general, nonlinear anisotropic shell for an explicit finite element environment. To our knowledge, this is the first shell capable of modeling experimentally driven strain energy functions for biologic tissue.

References

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