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Pricing American Options under the Constant Elasticity of Variance Model and Subject to Bankruptcy
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American-style Puts under the JDCEV Model: A Correction

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Abstract

Nunes (2009) prices American-style standard call and put options under the geometric Brownian motion, CEV, and JDCEV models, using an optimal stopping approach that is based on the first passage time density of the underlying asset price through the early exercise boundary. Under the JDCEV model, the solution provided by Nunes (2009, Equation 56) for the recovery component of the American-style put is wrong. This note corrects Nunes (2009, Equation 56), clarifies how the first hitting time density shall be determined, and compares the numerical results obtained with the *static hedge approach* recently offered by Ruas et al. (2013) for the JDCEV model.

I. Introduction

Nunes (2009) proposes an optimal stopping approach for the valuation of American-style options that is valid for any Markovian and diffusion underlying price process—such as the geometric Brownian motion or the CEV process—as well as for any parameterization of the (unknown) exercise boundary. The proposed approach only requires that the underlying price process provides a viable valuation method for European-style options and for its transition density function.

Nunes (2009, Section VII) extends this approach to the JDCEV framework of Carr and Linetsky (2006), where the price S of the underlying defaultable stock is modeled, under the equivalent martingale measure $\mathbb{Q}$, through the stochastic differential equation

$$\frac{dS_t}{S_t} = \left[r_t - q_t + \lambda\left(t, S\right)\right] dt + \sigma\left(t, S\right) dW_t^{\mathbb{Q}}, \tag{1}$$

with $S_{t_0} > 0$, and where the (short-term and risk-free) interest rate r_t and the dividend yield q_t are deterministic functions of time, $\sigma\left(t, S\right)$ represents the instantaneous volatility of equity returns, $\lambda\left(t, S\right)$ is the default intensity, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ is a standard Wiener process generating the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq t_0\}$. Carr and Linetsky (2006) assume that the stock price S can either diffuse or jump to default, i.e. the time of default is simply given by

$$\zeta = \tau_0 \wedge \tilde{\zeta}, \tag{2}$$

where $\tilde{\zeta}$ is the first jump time of a doubly-stochastic Poisson process with intensity $\lambda\left(t, S\right)$, while

$$\tau_0 := \inf\left\{t > t_0 : S_t = 0\right\} \tag{3}$$

is the first passage time (through diffusion) of the stock price to the bankruptcy level.

Under the JDCEV framework, Nunes (2009) decomposes the American-style option price into a similar contract that assumes no default (until the earliest between the maturity date and the early exercise date) and a recovery component. However, for the American-style put the solution provided by Nunes (2009, Equation 56) for such recovery component is wrong. Section II corrects the pricing solution of Nunes (2009, Equation 56) while Section

III identifies the nonlinear integral equation satisfied by the first hitting time density through the early exercise boundary and under the JDCEV model. Finally, Section IV summarizes the changes made to Nunes (2009, Section VII) and recomputes Nunes (2009, Table 5).

II. Corrected Pricing Solutions

Under the JDCEV model, and assuming that $\zeta > t_0$, the time-t_0 value of an American put (if $\phi = 1$) or call (if $\phi = -1$) on the stock price S, with strike price K, and with maturity date T ($\geq t_0$) is represented by Nunes (2009, Equation 53), i.e.:

$$V_{t_0}(S,K,T;\phi) = \sup_{\tau \in \mathcal{T}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{T\wedge\tau} r_l dl} (\phi K - \phi S_{T\wedge\tau})^+ \mathbb{1}_{\{\zeta > T\wedge\tau\}} \middle| \mathcal{G}_{t_0} \right] + \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_{t_0}^{\zeta} r_l dl} (\phi K)^+ \mathbb{1}_{\{\zeta \leq T\wedge\tau\}} \middle| \mathcal{G}_{t_0} \right] \right\}, \tag{4}$$

where $\mathcal{T}$ is the set of all stopping times (taking values in $[t_0, \infty]$) for the enlarged filtration $\mathbb{G} = \{\mathcal{G}_t : t \geq t_0\}$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$.[1]

The second term on the right-hand side of equation (4) is zero for an American-style call option; for an American put, it corresponds to a recovery payment equal to the strike K at the default time ζ (and as long as the default event precedes both the expiry and early exercise dates). The error in Nunes (2009) concerns the evaluation of this recovery value for the American-style put: Nunes (2009, Equation 56) wrongly assumes that $\{\zeta \leq T \wedge \tau\} = \{\zeta \leq T\}$, i.e. evaluates the recovery component associated not to an American-style but rather a European-style put option. However, the recovery component of an American-style put must be worth less than the one associated to the corresponding European-style contract because there can be no default after the (possible) early exercise event.[2]

To correct Nunes (2009, Equation 56), and following, for instance, Carr and Linetsky (2006, Equations 3.2 and 3.4), equation (4) can be rewritten in terms of the restricted

[1] $\mathbb{D} = \{\mathcal{D}_t : t \geq t_0\}$ is the filtration generated by the default indicator process $\mathcal{D}_t = \mathbb{1}_{\{t > \zeta\}}$.

[2] Note that Nunes (2009) assumes that the recovery value is paid at the default time but Carr and Linetsky (2006, Equation 3.11) considers also the possibility of recovery only at the maturity date of the option (for European-style puts). For American-style options, this latter case would be even simpler to model.

filtration $\mathbb{F}$ as long as the short-term interest rate is replaced by an intensity-adjusted short-rate:

$$(5)\quad V_{t_0}(S,K,T;\phi) = \sup_{\tau\in\mathcal{T}}\left\{\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau}(r_l+\lambda(l,S))dl}(\phi K-\phi S_{T\wedge\tau})^+\mathbb{1}_{\{\tau_0>T\wedge\tau\}}\middle|\mathcal{F}_{t_0}\right]\right.$$
$$\left.+(\phi K)^+\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T\wedge\tau}e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbb{1}_{\{\tau_0>v\}}dv\middle|\mathcal{F}_{t_0}\right]\right\}.$$

Moreover, since S behaves as a pure diffusion process with respect to the filtration $\mathbb{F}$, Detemple and Tian (2002, Propositions 1 and 2) show that there exists (at each time $t\in[t_0,T]$) a critical asset price E_t below (above) which the American-style put (call) price equals its intrinsic value and, therefore, early exercise should occur. Consequently, and representing the first passage time of the underlying asset price S to its early exercise boundary $\{E_t, t_0\le t\le T\}$ by Nunes (2009, Equation 5), i.e.

$$(6)\qquad \tau_e := \inf\{t\ge t_0 : S_t = E_t\},$$

equation (5) can be restated as:

$$(7)\qquad V_{t_0}(S,K,T;\phi) = V_{t_0}^0(S,K,T;\phi) + V_{t_0}^D(S,K,T;\phi),$$

where

$$(8)\qquad V_{t_0}^0(S,K,T;\phi) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau_e}(r_l+\lambda(l,S))dl}(\phi K-\phi S_{T\wedge\tau_e})^+\mathbb{1}_{\{\tau_0>T\wedge\tau_e\}}\middle|\mathcal{F}_{t_0}\right]$$

corresponds to Nunes (2009, Equation 55) i.e. to the American option price conditional on no default (before the expiry and early exercise dates), and

$$(9)\qquad V_{t_0}^D(S,K,T;\phi) = (\phi K)^+\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T\wedge\tau_e}e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbb{1}_{\{\tau_0>v\}}dv\middle|\mathcal{F}_{t_0}\right]$$

represents the present value of the recovery payment at the default time associated to the American-style put.

Nunes (2009, Equation 57 and Proposition 7) correctly decomposes the American option value conditional on no default (8) into the corresponding European option value (conditional on no default until the maturity date) and an early exercise premium, i.e.:

$$(10)\qquad V_{t_0}^0(S,K,T;\phi) = v_{t_0}^0(S,K,T;\phi) + EEP_{t_0}^0(S,K,T;\phi),$$

where

$$v_{t_0}^{0}(S,K,T;\phi) = e^{-\int_{t_0}^{T} r_l dl}\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^{+} \mathbb{1}_{\{\zeta > T\}} \middle| \mathcal{G}_{t_0} \right] \tag{11}$$

corresponds to Carr and Linetsky (2006, Equations 3.8 and 3.10), and

$$\begin{aligned} & EEP_{t_0}^{0}(S,K,T;\phi) \\ = & \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl}\left[(\phi K - \phi E_u)^{+} - v_u^{0}(E,K,T;\phi) \right] SP(t_0,u)\,\mathbb{Q}\left(\tau_e \in du | \mathcal{F}_{t_0}\right), \end{aligned} \tag{12}$$

with

$$SP(t_0,u) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u} \lambda(l,S)dl} \mathbb{1}_{\{\tau_0 > u\}} \middle| \mathcal{F}_{t_0} \right] \tag{13}$$

representing the risk-neutral probability of surviving beyond time $u > t_0$. However, the recovery component (9) must not be given by Nunes (2009, Equation 56) but, instead, must be also decomposed into an European-style recovery value and an additional (negative) early exercise premium component. Next proposition provides such decomposition and corrects Nunes (2009, Equation 56).

Proposition 1 *Under the JDCEV model, and assuming that $\zeta > t_0$, then*

$$V_{t_0}^{D}(S,K,T;\phi) = v_{t_0}^{D}(S,K,T;\phi) - EEP_{t_0}^{D}(S,K,T;\phi), \tag{14}$$

where

$$v_{t_0}^{D}(S,K,T;\phi) = (\phi K)^{+}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T} e^{-\int_{t_0}^{v}(r_l + \lambda(l,S))dl} \lambda(v,S) \mathbb{1}_{\{\tau_0 > v\}} dv \middle| \mathcal{F}_{t_0} \right] \tag{15}$$

is the recovery (at the default time) component of an European-style option contract, as given in Carr and Linetsky (2006, Equation 3.4), and

$$EEP_{t_0}^{D}(S,K,T;\phi) = \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl} v_u^{D}(E,K,T;\phi)\, SP(t_0,u)\,\mathbb{Q}\left(\tau_e \in du | \mathcal{F}_{t_0}\right) \tag{16}$$

is the early exercise component of the recovery value.

Proof. Since $\mathbb{1}_{\{\tau_e \geq T\}} = 1 - \mathbb{1}_{\{\tau_e < T\}}$, equation (9) can be rewritten as:

$$\begin{aligned} V_{t_0}^{D}(S,K,T;\phi) &= (\phi K)^{+}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T} e^{-\int_{t_0}^{v}(r_l + \lambda(l,S))dl} \lambda(v,S) \mathbb{1}_{\{\tau_0 > v, \tau_e \geq T\}} dv \middle| \mathcal{F}_{t_0} \right] \\ &\quad + (\phi K)^{+}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{\tau_e} e^{-\int_{t_0}^{v}(r_l + \lambda(l,S))dl} \lambda(v,S) \mathbb{1}_{\{\tau_0 > v, \tau_e < T\}} dv \middle| \mathcal{F}_{t_0} \right] \\ &= v_{t_0}^{D}(S,K,T;\phi) - EEP_{t_0}^{D}(S,K,T;\phi), \end{aligned}$$

i.e. as in equation (14), where

$$EEP^D_{t_0}(S,K,T;\phi) = (\phi K)^+ \mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T} e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbb{1}_{\{\tau_0>v,\tau_e<T\}}dv\middle|\mathcal{F}_{t_0}\right]$$

$$-(\phi K)^+ \mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{\tau_e} e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbb{1}_{\{\tau_0>v,\tau_e<T\}}dv\middle|\mathcal{F}_{t_0}\right]$$

$$= (\phi K)^+ \mathbb{E}_{\mathbb{Q}}\left[\int_{\tau_e}^{T} e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbb{1}_{\{\tau_0>v,\tau_e<T\}}dv\middle|\mathcal{F}_{t_0}\right]. \tag{17}$$

Taking advantage of the Markovian nature of the underlying price process S, the expectation on the right-hand side of equation (17) can be written as a convolution against the density of the first passage time τ_e, yielding

$$EEP^D_{t_0}(S,K,T;\phi)$$

$$= \int_{t_0}^{T}(\phi K)^+ \mathbb{E}_{\mathbb{Q}}\left[\int_{u}^{T} e^{-\int_{t_0}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbb{1}_{\{\tau_0>v\}}dv\mathbb{1}_{\{S_u=E_u\}}\middle|\mathcal{F}_{t_0}\right]\mathbb{Q}\left(\tau_e\in du|\mathcal{F}_{t_0}\right)$$

$$= (\phi K)^+\int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{u}\lambda(l,S)dl}\mathbb{1}_{\left\{\inf_{t_0<l\le u}(S_l)>0\right\}}\right.$$

$$\left.\mathbb{E}_{\mathbb{Q}}\left(\int_{u}^{T} e^{-\int_{u}^{v}(r_l+\lambda(l,S))dl}\lambda(v,S)\mathbb{1}_{\left\{\inf_{u\le l\le v}(S_l)>0\right\}}dv\middle|S_u=E_u\right)\middle|\mathcal{F}_{t_0}\right]\mathbb{Q}\left(\tau_e\in du|\mathcal{F}_{t_0}\right)$$

$$= \int_{t_0}^{T} e^{-\int_{t_0}^{u} r_l dl} SP(t_0,u)\, v^D_u(E,K,T;\phi)\,\mathbb{Q}\left(\tau_e\in du|\mathcal{F}_{t_0}\right),$$

where the last line corresponds to equation (16), and follows from equations (13) and (15). ∎

Since $\lambda(t,S)\ge 0$, for $t\in[t_0,T]$, Proposition 1 implies that

$$V^D_{t_0}(S,K,T;\phi)\le v^D_{t_0}(S,K,T;\phi).$$

Hence, and as expected, Nunes (2009, Equation 56) overvalues the recovery component of the American-style put since it simply corresponds to equation (15).

III. The First Passage Time Density

Under the geometric Brownian motion or the CEV process, the optimal stopping time τ_e is recovered by solving the nonlinear integral equation of Nunes (2009, Equation 35) through the standard partition method proposed by Park and Schuurmann (1976). However, under the JDCEV model, the early exercise event only occurs at the stopping time τ_e if $\tau_e < \zeta$. Therefore, under the JDCEV model, the density of the first passage time τ_e solves not Nunes (2009, Equation 35) but rather the following nonlinear integral equation.

Proposition 2 *Assuming that the underlying asset price S_t follows a JDCEV process and considering that the optimal exercise boundary is a continuous function of time, the first passage time density of the underlying asset price to the moving exercise boundary is the implicit solution of the following nonlinear integral equation:*

$$G_{-\phi}\left(t_0, S_{t_0}; u, E_u\right) = \int_{t_0}^{u} G_{-\phi}\left(v, E_v; u, E_u\right) \, SP\left(t_0, v\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right), \tag{18}$$

for $\phi S_{t_0} > \phi E_{t_0}$ and $u \in [t_0, T]$, and where $\phi = -1$ for an American call or $\phi = 1$ for an American put, with[3]

$$\begin{aligned} & G_{\phi}\left(v, E_v; u, E_u\right) \\ = \; & e^{-\int_v^u b_l dl} \left[\frac{k^2\left(v, v; E_v\right)}{\tau\left(v, u\right)}\right]^{\frac{1}{2|\bar{\beta}|}} \Phi_{\phi}\left(-\frac{1}{2|\bar{\beta}|}, \frac{k^2\left(v, u; E_u\right)}{\tau\left(v, u\right)}; 2\left(1 + v_+\right), \frac{k^2\left(v, v; E_v\right)}{\tau\left(v, u\right)}\right), \end{aligned} \tag{19}$$

$$v_+ := \frac{c + \frac{1}{2}}{|\bar{\beta}|}, \tag{20}$$

$$\tau\left(v, u\right) := \int_v^u a_s^2 e^{-2|\bar{\beta}| \int_v^s \alpha_l dl} ds, \tag{21}$$

$$k\left(v, u; S\right) := \frac{1}{|\bar{\beta}|} S^{|\bar{\beta}|} e^{-|\bar{\beta}| \int_v^u \alpha_l dl} \tag{22}$$

and

$$\alpha_l := r_l - q_l + b_l. \tag{23}$$

Proof. By the *law of total probability*,

$$\mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u \middle| \mathcal{G}_{t_0}\right) = \mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u, \tau_e \leq u \middle| \mathcal{G}_{t_0}\right), \tag{24}$$

[3] The function $\Phi_{\phi}\left(p, y; v, \lambda\right) := \mathbb{E}^{\chi^2(v,\lambda)}\left(X^p \mathbb{1}_{\{\phi X \geq \phi y\}}\right)$ represents, for $\phi \in \{-1, 1\}$, the truncated p-th moments of a noncentral chi-square random variable X with v degrees of freedom and noncentrality parameter λ, as defined in Carr and Linetsky (2006, Equations 5.11 and 5.12).

since $\mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u, \tau_e > u \middle| \mathcal{G}_{t_0}\right) = 0$.

Starting with the left-hand side of equation (24) and using Carr and Linetsky (2006, Equation 3.2), then

$$\begin{aligned}\mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u \middle| \mathcal{G}_{t_0}\right) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\phi S_u \leq \phi E_u, \zeta > u\}} \middle| \mathcal{G}_{t_0}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^u \lambda(S,l)dl} \mathbb{1}_{\{\phi S_u \leq \phi E_u, \tau_0 > u\}} \middle| \mathcal{F}_{t_0}\right). \end{aligned} \tag{25}$$

Moreover, Carr and Linetsky (2006, Proposition 5.4) allows equation (25) to be rewritten as

$$\begin{aligned} &\mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u \middle| \mathcal{G}_{t_0}\right) \\ = \; & e^{-\int_{t_0}^u b_l dl} \mathbb{E}^{(v_+)}_{k\left(t_0,t_0;S_{t_0}\right)} \left[\left(\frac{R_{\tau(t_0,u)}}{k\left(t_0,t_0;S_{t_0}\right)} \right)^{-\frac{1}{|\bar{\beta}|}} \mathbb{1}_{\left\{\phi e^{\int_{t_0}^u \alpha_s ds} \left(\left|\bar{\beta}\right| R_{\tau(t_0,u)}\right)^{\frac{1}{|\bar{\beta}|}} \leq \phi E_u\right\}} \right] \\ = \; & e^{-\int_{t_0}^u b_l dl} \frac{\mathbb{E}^{(v_+)}_{k\left(t_0,t_0;S_{t_0}\right)} \left[\left(\frac{R^2_{\tau(t_0,u)}}{\tau(t_0,u)} \right)^{-\frac{1}{2|\bar{\beta}|}} \mathbb{1}_{\left\{\phi \frac{R^2_{\tau(t_0,u)}}{\tau(t_0,u)} \leq \phi \frac{E_u^{2|\bar{\beta}|} e^{-2|\bar{\beta}| \int_{t_0}^u \alpha_s ds}}{|\bar{\beta}|^2 \tau(t_0,u)}\right\}} \right]}{\left[\frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\tau(t_0,u)}\right]^{-\frac{1}{2|\bar{\beta}|}}}, \end{aligned} \tag{26}$$

where the expectation is taken with respect to the law of a Bessel process $\left\{R_{\tau(t_0,u)}; u \geq t_0\right\}$ of index v_+ and started at $R_{\tau(t_0,t_0)} = k\left(t_0, t_0; S_{t_0}\right)$. Finally, using Carr and Linetsky (2006, Equations 5.11 and 5.12), and since the process $\frac{R^2_{\tau(t_0,u)}}{\tau(t_0,u)}$ follows a noncentral chi-square law with $2\left(1+v_+\right)$ degrees of freedom and non-centrality parameter $\frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\tau(t_0,u)}$, equation (26) yields

$$\begin{aligned}\mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u \middle| \mathcal{G}_{t_0}\right) &= e^{-\int_{t_0}^u b_l dl} \frac{\Phi_{-\phi}\left(-\frac{1}{2|\bar{\beta}|}, \frac{k^2(t_0,u;E_u)}{\tau(t_0,u)}; 2\left(1+v_+\right), \frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\tau(t_0,u)}\right)}{\left[\frac{k^2\left(t_0,t_0;S_{t_0}\right)}{\tau(t_0,u)}\right]^{-\frac{1}{2|\bar{\beta}|}}} \\ &= G_{-\phi}\left(t_0, S_{t_0}; u, E_u\right). \end{aligned} \tag{27}$$

Concerning the right-hand side of equation (24),

$$\begin{aligned}\mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u, \tau_e \leq u \middle| \mathcal{G}_{t_0}\right) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{\phi S_u \leq \phi E_u, \zeta > u, \tau_e \leq u\}} \middle| \mathcal{G}_{t_0}\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^u \lambda(S,l)dl} \mathbb{1}_{\{\phi S_u \leq \phi E_u, \tau_0 > u, \tau_e \leq u\}} \middle| \mathcal{F}_{t_0}\right), \end{aligned} \tag{28}$$

where the last line follows again from Carr and Linetsky (2006, Equation 3.2). Since S_t behaves as a pure and Markovian diffusion process with respect to the filtration $\mathbb{F}$, equation

(28) can be rewritten in terms of the convolution between the densities of the first passage time τ_e and of the random vector (S_u, τ_0):

$$
\begin{aligned}
& \mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u, \tau_e \leq u \middle| \mathcal{G}_{t_0}\right) \\
= & \int_{t_0}^{u} \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{t_0}^{u} \lambda(S,l)dl} \mathbb{1}_{\{\phi S_u \leq \phi E_u, \tau_0 > u, S_v = E_v\}} \middle| \mathcal{F}_{t_0}\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) \\
= & \int_{t_0}^{u} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{v}^{u} \lambda(S,l)dl} \mathbb{1}_{\{\phi S_u \leq \phi E_u, \inf_{v \leq l \leq u}(S_l) > 0\}} \middle| S_v = E_v\right) \right. \\
& \left. e^{-\int_{t_0}^{v} \lambda(S,l)dl} \mathbb{1}_{\{\inf_{t_0 \leq l \leq v}(S_l) > 0\}} \middle| \mathcal{F}_{t_0}\right] \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right),
\end{aligned}
\tag{29}
$$

where the last equality follows from the *tower law for conditional expectations.* Comparing the inner expectation on the right-hand side of equation (29) with equation (25), and using equation (27), then

$$
\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_{v}^{u} \lambda(S,l)dl} \mathbb{1}_{\{\phi S_u \leq \phi E_u, \inf_{v \leq l \leq u}(S_l) > 0\}} \middle| S_v = E_v\right) = G_{-\phi}\left(v, E_v; u, E_u\right),
$$

and, therefore, equation (29) becomes

$$
\begin{aligned}
& \mathbb{Q}\left(\phi S_u \leq \phi E_u, \zeta > u, \tau_e \leq u \middle| \mathcal{G}_{t_0}\right) \\
= & \int_{t_0}^{u} G_{-\phi}\left(v, E_v; u, E_u\right) \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{v} \lambda(S,l)dl} \mathbb{1}_{\{\inf_{t_0 \leq l \leq v}(S_l) > 0\}} \middle| \mathcal{F}_{t_0}\right] \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) \\
= & \int_{t_0}^{u} G_{-\phi}\left(v, E_v; u, E_u\right) SP\left(t_0, v\right) \mathbb{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right),
\end{aligned}
\tag{30}
$$

where the last line follows from definition (13).

Combining equations (24), (27), and (30), equation (18) arises immediately. ∎

IV. Changes to the Original Paper

In summary, the first correction that shall be done to the pricing solutions proposed in Nunes (2009, Section VII) is to replace Nunes (2009, Equation 56) by equation (14). Combining equations (7), (10), and (14), this is equivalent to rewrite the American-style option price under the JDCEV model as

$$
V_{t_0}\left(S, K, T; \phi\right) = \left[v_{t_0}^{0}\left(S, K, T; \phi\right) + v_{t_0}^{D}\left(S, K, T; \phi\right)\right] + EEP_{t_0}\left(S, K, T; \phi\right), \tag{31}
$$

where

$$EEP_{t_0}(S,K,T;\phi) = EEP^0_{t_0}(S,K,T;\phi) - EEP^D_{t_0}(S,K,T;\phi) \tag{32}$$

represents the overall early exercise premium while the first two terms on the right-hand side of equation (31) are simply the price of an European-style option whose recovery component assumes payment at the default time.

Since the time path $\{E_t, t_0 \leq t \leq T\}$ of critical asset prices is not known ex ante, to use equation (31), we must first parameterize such early exercise boundary, and maximize (with respect to those parameters) not Nunes (2009, Equation 60) but the overall early exercise premium (32). For this purpose, and as a second correction, the density of the optimal passage time τ_e shall be recovered by solving—through the standard partition method proposed by Park and Schuurmann (1976)—the nonlinear integral equation (18) instead of Nunes (2009, Equation 35).

Using the above two mentioned corrections, Table 1 recomputes Nunes (2009, Table 5).

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Table 1: American Put Options under the JDCEV Model, with $\bar{\beta} = -0.5$, $c = 0.5$, $b = 0.02$, $S_{t_0} = \$100$ and $T - t_0 = 5$ years

Option Parameters	Strike	European Put $v^0_{t_0}$	European Put $v^D_{t_0}$	European Put v_{t_0}	American Put, Nunes (2009) $EEP^0_{t_0}$	Nunes (2009) $V^0_{t_0}$	Nunes (2009) $EEP^D_{t_0}$	Nunes (2009) $V^D_{t_0}$	Nunes (2009) V_{t_0}	SHP V_{t_0}
	80	1.095	11.799	12.893	0.693	1.787	0.468	11.331	13.119	13.119
$r = 7\%$	90	1.842	13.274	15.116	1.432	3.274	0.951	12.322	15.596	15.597
$q = 3\%$	100	2.895	14.749	17.644	2.774	5.670	1.823	12.925	18.595	18.596
$a = 2$	110	4.302	16.223	20.526	5.101	9.404	3.326	12.897	22.301	22.302
	120	6.098	17.698	23.797	8.989	15.087	5.837	11.861	26.948	26.950
	80	1.797	26.818	28.615	2.809	4.606	2.585	24.234	28.840	28.845
$r = 7\%$	90	2.504	30.171	32.675	4.128	6.632	3.748	26.422	33.054	33.059
$q = 3\%$	100	3.358	33.523	36.881	5.845	9.203	5.239	28.284	37.487	37.489
$a = 4$	110	4.363	36.875	41.239	8.038	12.402	7.113	29.762	42.164	42.166
	120	5.525	40.227	45.753	10.796	16.322	9.435	30.792	47.114	47.116
	80	1.400	17.688	19.088	1.624	3.024	1.287	16.401	19.425	19.426
$r = 7\%$	90	2.054	19.899	21.953	2.690	4.743	2.082	17.818	22.561	22.562
$q = 0\%$	100	2.884	22.110	24.994	4.291	7.174	3.252	18.858	26.033	26.034
$a = 3$	110	3.906	24.321	28.227	6.636	10.542	4.938	19.383	29.925	29.927
	120	5.134	26.532	31.666	9.991	15.124	7.313	19.219	34.343	34.345
	80	5.385	22.785	28.170	1.312	6.697	1.273	21.512	28.209	28.209
$r = 3\%$	90	7.496	25.633	33.129	1.849	9.345	1.787	23.845	33.191	33.191
$q = 7\%$	100	10.001	28.481	38.482	2.516	12.518	2.423	26.058	38.575	38.575
$a = 3$	110	12.891	31.329	44.220	3.332	16.223	3.198	28.131	44.354	44.354
	120	16.148	34.177	50.325	4.314	20.462	4.126	30.051	50.513	50.513

Table 1 values American-style put options under a time-homogeneous jump to default extended CEV (JDCEV) model. The third column contains European-style put prices, conditional on no default, as given by equation (11). The fourth column reports the recovery component of the European put contract, which is computed from Carr and Linetsky (2006, Equation 5.15), and the fifth column yields the sum of the previous two components. The early exercise premium components (sixth and eighth columns) are computed through equations (12) and (16) and via Proposition 2, using a polynomial specification for the exercise boundary with four degrees of freedom. The seventh, ninth and tenth columns are given by equations (10), (14) and (7), respectively. For comparison purposes, the last column presents the prices of the American-style put options computed through the *static hedge approach* proposed by Ruas et al. (2013).