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Appendices

to "Bullish/Bearish Strategies of Trading: A Nonlinear Equilibrium"

A Proof of proposition 1

The informed agent i maximizes his expected profit conditionally on the observed signal S. If the informed agent i observes the value $S = 0$ (which occurs with probability 0), it is straightforward to show that he optimally submits 0. Given the structure of his information and the symmetry of the problem, his optimal strategy is of the form:

$$\widetilde{x}_i = \gamma_i, \quad if \ S = 1,$$

$$\widetilde{x}_i = -\gamma_i, \ if \ S = -1\cdot$$

Given their prior, the market makers conjecture that the informed agent i submits the following order:

$$\widetilde{x} = \gamma_i^c, \quad if \ S = 1,$$

$$\widetilde{x}_i = -\gamma_i^c, \ if \ S = -1,$$

The market makers set prices according to the following probabilities :

$$\widetilde{p} = E[\widetilde{v}/w] = \frac{2\sigma_v}{\sqrt{2\pi}}[P_u(w,N) - P_d(w,N)] = \frac{2\sigma_v}{\sqrt{2\pi}}\tanh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right), \tag{1}$$

$$P_u(w,N) = \frac{\varphi\left(\frac{w-\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)}{\varphi\left(\frac{w-\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)+\varphi\left(\frac{w+\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)} = \frac{\cosh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)+\sinh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)}{2\cosh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)} = \frac{1}{2}\left(1+\tanh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)\right),$$

$$P_d(w,N) = \frac{\varphi\left(\frac{w+\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)}{\varphi\left(\frac{w-\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)+\varphi\left(\frac{w+\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)} = \frac{\cosh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)-\sinh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)}{2\cosh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)} = \frac{1}{2}\left(1-\tanh\left(\frac{w\sum\limits_{i=1}^{N}\gamma_i^c}{\sigma_u^2}\right)\right),$$

$where:$

$\bullet \varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right),$

$\bullet P_u(w, N) + P_d(w, N) = 1.$

The proof proceeds in two steps. We first compute the joint density of $(\widetilde{v}, \widetilde{w}) : f_{\widetilde{v},\widetilde{w}}$ and therefore the conditional probability density function of $\widetilde{v}$ given w. We thus deduce the conditional expectation $E[\widetilde{v}/w]$. Moreover we define $\Phi(x) = Prob(\widetilde{x} \leq x)$ with $\widetilde{x} \rightsquigarrow \mathcal{N}(0, 1)$.

$$F_{\widetilde{v},\widetilde{w}}(v, w) = Prob[\widetilde{v} \leq v, \widetilde{w} \leq w],$$

$$= Prob[\widetilde{v} \leq v, \widetilde{w} \leq w/\widetilde{v} < 0]Prob[\widetilde{v} < 0] + Prob[\widetilde{v} \leq v, \widetilde{w} \leq w/\widetilde{v} > 0]Prob[\widetilde{v} > 0],$$

$$= Prob\left[\widetilde{v} \leq min(0, v), -\sum_{i=1}^{N} \gamma_i^c + \widetilde{u} \leq w\right] + Prob\left[0 < \widetilde{v} \leq v, \sum_{i=1}^{N} \gamma_i^c + \widetilde{u} \leq w\right],$$

$$= \Phi\left(\frac{w + \sum_{i=1}^{N} \gamma_i^c}{\sigma_u}\right)\left[\Phi\left(\frac{v}{\sigma_v}\right) 1\!\!I_{v \leq 0} + \frac{1}{2} 1\!\!I_{v>0}\right] + \Phi\left(\frac{w - \sum_{i=1}^{N} \gamma_i^c}{\sigma_u}\right)\left[\Phi\left(\frac{v}{\sigma_v}\right) 1\!\!I_{v>0} - \frac{1}{2} 1\!\!I_{v>0}\right],$$

If $v < 0$:

$$F_{\widetilde{v},\widetilde{w}}(v, w) = \Phi\left(\frac{v}{\sigma_v}\right)\Phi\left(\frac{w + \sum_{i=1}^{N} \gamma_i^c}{\sigma_u}\right),$$

$$f_{\widetilde{v},\widetilde{w}}(v, w) = \frac{1}{\sigma_u \sigma_v}\varphi\left(\frac{v}{\sigma_v}\right)\varphi\left(\frac{w + \sum_{i=1}^{N} \gamma_i^c}{\sigma_u}\right).$$

If $v > 0$:

$$F_{\widetilde{v},\widetilde{w}}(v,w) = \Phi\left(\frac{v}{\sigma_v}\right)\Phi\left(\frac{w-\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right) + \frac{1}{2}\left[\Phi\left(\frac{w+\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right) - \Phi\left(\frac{w-\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)\right],$$

$$f_{\widetilde{v},\widetilde{w}}(v,w) = \frac{1}{\sigma_u\sigma_v}\varphi\left(\frac{v}{\sigma_v}\right)\varphi\left(\frac{w-\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right).$$

therefore : $f_{\widetilde{w}}(w) = \dfrac{1}{2\sigma_u}\left[\varphi\left(\dfrac{w+\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right) + \varphi\left(\dfrac{w-\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)\right]$

We also have, applying Bayes rule, that:

$$f_{\widetilde{v}/\widetilde{w}}(v,w) = \frac{f_{\widetilde{v},\widetilde{w}}(v,w)}{f_{\widetilde{w}}(w)},$$

and thus:

- $if\ \ v < 0,\quad f_{\widetilde{v}/\widetilde{w}}(v,w) = \dfrac{2}{\sigma_v}\dfrac{\varphi\left(\frac{v}{\sigma_v}\right)\varphi\left(\frac{w+\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)}{\varphi\left(\frac{w-\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)+\varphi\left(\frac{w+\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)} = \dfrac{2}{\sigma_v}\varphi\left(\frac{v}{\sigma_v}\right)P_d(w,N),$

- $if\ \ v > 0,\quad f_{\widetilde{v}/\widetilde{w}}(v,w) = \dfrac{2}{\sigma_v}\dfrac{\varphi\left(\frac{v}{\sigma_v}\right)\varphi\left(\frac{w-\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)}{\varphi\left(\frac{w-\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)+\varphi\left(\frac{w+\sum_{i=1}^{N}\gamma_i^c}{\sigma_u}\right)} = \dfrac{2}{\sigma_v}\varphi\left(\frac{v}{\sigma_v}\right)P_u(w,N).$

Then : $E[\widetilde{v}/w] = \displaystyle\int_{-\infty}^{0^-}\frac{2v}{\sigma_v}\varphi\left(\frac{v}{\sigma_v}\right)P_d(w,N) + \int_{0^+}^{+\infty}\frac{2v}{\sigma_v}\varphi\left(\frac{v}{\sigma_v}\right)P_u(w,N)dv,$

Which can be rewritten :
$E[\widetilde{v}/w] = \dfrac{2\sigma_v}{\sqrt{2\pi}}\left(P_u(w,N) - P_d(w,N)\right) = \dfrac{2\sigma_v}{\sqrt{2\pi}}\tanh\left(\dfrac{w\sqrt{N}}{\sigma_u}\right) = \widetilde{p}(w,N).$

We define :
$\gamma = (\gamma_1, \ldots, \gamma_N)'$ and $\gamma_c = (\gamma_1^c, \ldots, \gamma_N^c)'$.

The expected profit $\pi_i(\gamma, \gamma_c)$ of agent i is:

$$\pi_i(\gamma, \gamma_c) = \frac{1}{2} \left\{ E[\gamma_i(\widetilde{v} - \widetilde{p}(w, N))/\widetilde{v} > 0] + E[-\gamma_i(\widetilde{v} - \widetilde{p}(w, N))/\widetilde{v} < 0] \right\},$$

$$= \frac{1}{2} \left\{ \int_0^{+\infty} \int_{-\infty}^{+\infty} \gamma_i (v - \widetilde{p}(w, N)) \tfrac{2}{\sigma_u \sigma_v} \varphi\left(\tfrac{v}{\sigma_v}\right) \varphi\left(\frac{w - \sum\limits_{i=1}^{N} \gamma_i}{\sigma_u}\right) dvdw \right.$$

$$\left. + \int_{-\infty}^{0} \int_{-\infty}^{+\infty} - \gamma_i (v - \widetilde{p}(w, N)) \tfrac{2}{\sigma_u \sigma_v} \varphi\left(\tfrac{v}{\sigma_v}\right) \varphi\left(\frac{w + \sum\limits_{i=1}^{N} \gamma_i}{\sigma_u}\right) dvdw \right\}.$$

We make the following change in variables for the second integral $\begin{pmatrix} v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} -v \\ -w \end{pmatrix}$

while taking into account that $\varphi(x) = \varphi(-x)$ and $\widetilde{p}(-w, N) = -\widetilde{p}(w, N)$.

Moreover, for simplifications we set :

$$s(\gamma^c) = \frac{\sum_{i=1}^{N} \gamma_i^c}{\sigma_u},$$

$$s(\gamma) = \frac{\sum_{i=1}^{N} \gamma_i}{\sigma_u},$$

then : $\pi_i(s(\gamma), s(\gamma^c)) = \dfrac{2\gamma_i}{\sigma_u \sigma_v} \displaystyle\int_0^{+\infty} \int_{-\infty}^{+\infty} (v - \widetilde{p}(w, N)) \varphi\left(\tfrac{v}{\sigma_v}\right) \varphi\left(\tfrac{w}{\sigma_u} - s(\gamma)\right) dvdw,$

$$= \frac{2\gamma_i}{\sigma_u \sigma_v} \int_0^{+\infty} \int_{-\infty}^{+\infty} (v + p^* - 2p^* P_u(w, N)) \varphi\left(\tfrac{v}{\sigma_v}\right) \varphi\left(\tfrac{w}{\sigma_u} - s(\gamma)\right) dvdw,$$

$$= \frac{2\gamma_i}{\sigma_u \sigma_v} \left(\int_0^{+\infty} \int_{-\infty}^{+\infty} \varphi\left(\frac{v}{\sigma_v}\right) \left((v + p^*) \varphi\left(\frac{w}{\sigma_u} - s(\gamma)\right) - 2p^* P_u(w, N) \varphi\left(\frac{w}{\sigma_u} - s(\gamma)\right) \right) dvdw \right),$$

$$= \frac{2\gamma_i}{\sigma_u \sigma_v} \left(\frac{2\sigma_u \sigma_v^2}{\sqrt{2\pi}} - \frac{2\sigma_v^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(P_u(w, N) \varphi\left(\frac{w}{\sigma_u} - s(\gamma)\right) dw \right) \right),$$

$$= \frac{4\gamma_i \sigma_v}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} \left(\frac{1}{\sigma_u} \frac{\varphi\left(\frac{w}{\sigma_u} + s(\gamma^c)\right)}{\varphi\left(\frac{w}{\sigma_u} - s(\gamma^c)\right) + \varphi\left(\frac{w}{\sigma_u} + s(\gamma^c)\right)} \varphi\left(\frac{w}{\sigma_u} - s(\gamma)\right) \right) dw \right),$$

We now define the change in variables $t = \frac{w - \sum_{i=1}^{N} \gamma_i}{\sigma_u} = \frac{w}{\sigma_u} - s(\gamma)$, with $dt = \frac{dw}{\sigma_u}$, and get :

$$\pi_i(s(\gamma), s(\gamma^c)) = \frac{4\gamma_i \sigma_v}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\varphi(t + s(\gamma) + s(\gamma^c))}{\varphi(t + s(\gamma) - s(\gamma^c)) + \varphi(t + s(\gamma) + s(\gamma^c))} \varphi(t)\, dt,$$

$$\pi_i(s(\gamma), s(\gamma^c)) = \frac{2\gamma_i \sigma_v}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}(t + 2s(\gamma^c))^2\right) \exp\left(2s(\gamma^c)(s(\gamma) - s(\gamma^c))\right)}.$$

At the value $\gamma_i = 0$ and when γ_i tends towards infinity the value of the expected profit is zero, therefore there exists an interior solution $\gamma_i^*(\gamma_{-i}, \gamma_c)$ which maximises the expected profit. This solution must verify the first order condition $\frac{\partial \pi_i}{\partial \gamma_i}(\gamma, \gamma_c) = 0$.

$$\frac{\partial \pi_i}{\partial \gamma_i}(\gamma, \gamma_c) = \frac{2\sigma_v}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}(t + 2s(\gamma^c))^2\right) \exp\left(2s(\gamma^c)(s(\gamma) - s(\gamma^c))\right)}$$

$$-\frac{\gamma_i}{\sigma_u} \frac{2\sigma_v}{\pi} \int_{-\infty}^{+\infty} \frac{2s(\gamma^c) \exp\left(\frac{1}{2}(t + 2s(\gamma^c))^2\right) \exp\left(2s(\gamma^c)(s(\gamma) - s(\gamma^c))\right)}{\left(\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}(t + 2s(\gamma^c))^2\right) \exp\left(2s(\gamma^c)(s(\gamma) - s(\gamma^c))\right)\right)^2} dt.$$

Each agent i chooses γ_i to maximise his expected profit, and therefore at the equilibrium we have for $i = 1, \ldots, N$:

$$\frac{\partial \pi_i}{\partial \gamma_i}(\gamma, \gamma_c) = 0, \tag{2}$$

$$\gamma_i^c = \gamma_i, \tag{3}$$

This implies that:

$$0 = \int_{-\infty}^{+\infty} \frac{dt}{\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}(t + 2s(\gamma^c))^2\right) \exp\left(2s(\gamma^c)(s(\gamma) - s(\gamma^c))\right)}$$

$$-\frac{\gamma_i}{\sigma_u} \int_{-\infty}^{+\infty} \frac{2s(\gamma^c) \exp\left(\frac{1}{2}(t + 2s(\gamma^c))^2\right) \exp\left(2s(\gamma^c)(s(\gamma) - s(\gamma^c)\right)}{\left(\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}(t + 2s(\gamma^c))^2\right) \exp\left(2s(\gamma^c)(s(\gamma) - s(\gamma^c))\right)\right)^2} dt.$$

Thus, for all $i = 1, \ldots, N$ we have :

$$\gamma_i = \sigma_u \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{dt}{\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}\left(t + 2s\left(\gamma\right)\right)^2\right)}}{\displaystyle\int_{-\infty}^{+\infty} \frac{2s\left(\gamma\right)\exp\left(\frac{1}{2}\left(t + 2s\left(\gamma\right)\right)^2\right) dt}{\left[\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}\left(t + 2s\left(\gamma\right)\right)^2\right)\right]^2}}. \tag{4}$$

We have proved, at the equilibrium, that the first order conditions imply that all agents have the same strategy, in other words, there exists $\gamma^*(N)$ independent of i such as for all $i = 1, \ldots, N$ we have :

$$\gamma_i = \gamma^*(N). \tag{5}$$

Let $H(\cdot)$ and $K(\cdot)$ be two functions defined by :

$$H(x) = \int_{-\infty}^{+\infty} \frac{dt}{\exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}(t + 2x)^2\right)}$$

and

$$K(x) = \int_{-\infty}^{+\infty} \frac{2x\exp\left(\frac{1}{2}(t + 2x)^2\right) dt}{\left[\exp(\frac{1}{2}t^2) + \exp\left(\frac{1}{2}(t + 2x)^2\right)\right]^2}.$$

For all x it can be shown that :

$$K(x) = xH(x) \tag{6}$$

Using the definitions of $H(\cdot)$ and $K(\cdot)$ and using (6) imply that

$$H\left(\frac{N\gamma^*(N)}{\sigma_u}\right) - \frac{\sigma_u}{N\gamma^*(N)} K\left(\frac{N\gamma^*(N)}{\sigma_u}\right) = 0,$$

Reorganizing terms using (3), (4) and (5) we get :

$$H\left(\frac{N\gamma^*(N)}{\sigma_u}\right)\left(1-\frac{\sigma_u}{N\gamma^*(N)}\frac{\sigma_u}{\gamma^*(N)}\right)=0,\ \ .$$

Finally $\gamma^*(N)=\frac{\sigma_u}{\sqrt{N}}$.

Concerning the second order conditions we need to prove that $\frac{\partial^2\pi}{\partial\gamma_i^2}\left(\frac{\sigma_u}{\sqrt{N}},\ldots,\frac{\sigma_u}{\sqrt{N}}\right)<0$.

$$\frac{\partial^2\pi}{\partial\gamma_i^2}\left(\gamma,\gamma_c\right)=\frac{2\sigma_v}{\pi}\left\{\frac{-4s\left(\gamma^c\right)}{\sigma_u}\int_{-\infty}^{+\infty}\frac{\exp\left(\frac{1}{2}\left(t+2s\left(\gamma^c\right)\right)^2\right)\exp\left(2s\left(\gamma^c\right)\left(s\left(\gamma\right)-s\left(\gamma^c\right)\right)\right)dt}{\left[\exp\left(\frac{1}{2}t^2\right)+\exp\left(\frac{1}{2}\left(t+2s\left(\gamma^c\right)\right)^2\right)\exp\left(2s\left(\gamma^c\right)\left(s\left(\gamma\right)-s\left(\gamma^c\right)\right)\right)\right]^2}-\right.$$

$$4\gamma_i\frac{\left(s(\gamma^c)\right)^2}{\sigma_u^2}\int_{-\infty}^{+\infty}\frac{\exp\left(\left(t+2s\left(\gamma^c\right)\right)^2\right)\exp\left(4s\left(\gamma^c\right)\left(s\left(\gamma\right)-s\left(\gamma^c\right)\right)\right)dt}{\left[\exp\left(\frac{1}{2}t^2\right)+\exp\left(\frac{1}{2}\left(t+2s\left(\gamma^c\right)\right)^2\right)\exp\left(2s\left(\gamma^c\right)\left(s\left(\gamma\right)-s\left(\gamma^c\right)\right)\right)\right]^2}+$$

$$\left.8\gamma_i\frac{\left(s(\gamma^c)\right)^2}{\sigma_u^2}\int_{-\infty}^{+\infty}\frac{\exp\left(\left(t+2s\left(\gamma^c\right)\right)^2\right)\exp\left(4s\left(\gamma^c\right)\left(s\left(\gamma\right)-s\left(\gamma^c\right)\right)\right)dt}{\left[\exp\left(\frac{1}{2}t^2\right)+\exp\left(\frac{1}{2}\left(t+2s\left(\gamma^c\right)\right)^2\right)\exp\left(2s\left(\gamma^c\right)\left(s\left(\gamma\right)-s\left(\gamma^c\right)\right)\right)\right]^3}\right\},$$

$$\frac{\partial^2\pi}{\partial^2\gamma_i}\left(\gamma^*(N),\gamma^*(N)\right)=-\frac{16\sigma_v\sqrt{N}}{\sigma_u\pi}\int_{-\infty}^{+\infty}\frac{\exp\left(\frac{1}{2}\left(t+2\sqrt{N}\right)^2\right)\exp\left(\frac{1}{2}t^2\right)dt}{\left[\exp\left(\frac{1}{2}t^2\right)+\exp\left(\frac{1}{2}\left(t+2\sqrt{N}\right)^2\right)\right]^3}<0\cdot$$

This ends both proofs of existence and uniqueness.

$$\pi_i\left(\gamma^*(N),\gamma^*(N)\right)=\frac{2\sigma_u\sigma_v}{\pi\sqrt{N}}\int_{-\infty}^{+\infty}\frac{dt}{\exp\left(\frac{1}{2}t^2\right)+\exp\left(\frac{1}{2}\left(t+2\sqrt{N}\right)^2\right)}.$$

B Proof of propositions 2 and 3

Profits:

The individual expected profit is given by:

$$\begin{aligned}\pi^*(\gamma^*(N),\gamma^*(N)) &= \frac{2\sigma_u\sigma_v}{\pi\sqrt{N}}\int_{-\infty}^{+\infty}\frac{dt}{\exp\left(\frac{1}{2}t^2\right)+\exp\left(\frac{1}{2}\left(t+2\sqrt{N}\right)^2\right)}\\
&= \frac{2\sigma_u\sigma_v}{\pi\sqrt{N}}\int_{-\infty}^{+\infty}\frac{dt}{\exp\left(\frac{1}{2}\left(t-\sqrt{N}\right)^2\right)+\exp\left(\frac{1}{2}\left(t+\sqrt{N}\right)^2\right)}\\
&= \frac{4\sigma_u\sigma_v}{\pi\sqrt{N}}\int_{0}^{+\infty}\frac{dt}{\exp\left(\frac{1}{2}\left(t-\sqrt{N}\right)^2\right)+\exp\left(\frac{1}{2}\left(t+\sqrt{N}\right)^2\right)}\\
&= \frac{4\sigma_u\sigma_v}{\pi\sqrt{N}}\int_{0}^{+\infty}\frac{\exp\left(-\frac{1}{2}\left(t+\sqrt{N}\right)^2\right)dt}{1+\exp\left(-2t\sqrt{N}\right)} = \frac{4\sigma_u\sigma_v}{\pi\sqrt{N}}\theta\left(\sqrt{N}\right)\end{aligned}$$

with $\theta(x) = \int_0^{+\infty}\frac{\exp\left(-\frac{1}{2}(t+x)^2\right)}{1+\exp(-2tx)}dt$. For $x>0$, we perform the following changes in variables $u = tx$.

$$\theta(x) = \frac{1}{x}\int_0^{+\infty}\frac{\exp\left[-\frac{1}{2}\left(\frac{u}{x}+x\right)^2\right]}{1+\exp(-2u)}du,$$

$$\theta(x) = \frac{1}{x}\exp\left(-\frac{1}{2}x^2\right)\int_0^{+\infty}\frac{\exp(-u)\exp\left(-\frac{1}{2}\frac{u^2}{x^2}\right)}{1+\exp(-2u)}du\cdot$$

If $u>0$ then $0<\frac{\exp(-u)}{1+\exp(-2u)}<1$ and we get :

$$0<\theta(x)<\frac{1}{x}\exp\left(-\frac{1}{2}x^2\right)\int_0^{+\infty}\exp\left(-\frac{1}{2}\frac{u^2}{x^2}\right)du = \frac{1}{x}\exp\left(-\frac{1}{2}x^2\right)\frac{1}{2}x\sqrt{2\pi} = \frac{\sqrt{2\pi}}{2}\exp\left(-\frac{1}{2}x^2\right),$$

thus :

$$0<\pi^*(\gamma^*(N),\gamma^*(N))<\frac{4\sigma_u\sigma_v}{\pi\sqrt{N}}\frac{\sqrt{2\pi}}{2}\exp\left(-\frac{1}{2}N\right) = \frac{2\sqrt{2}\sigma_u\sigma_v}{\sqrt{\pi}}\frac{1}{\sqrt{N}}\exp\left(-\frac{N}{2}\right).$$

In other words, at the limit we have :

(7) $$\lim_{N\to+\infty}\pi^*(\gamma*(N),\gamma^*(N)) = \lim_{N\to+\infty}N\frac{2\sqrt{2}\sigma_u\sigma_v}{\sqrt{\pi}}\frac{1}{\sqrt{N}}\exp\left(-\frac{N}{2}\right) = 0$$

(8) $$\lim_{N\to+\infty}N\pi^*(\gamma*(N),\gamma^*(N)) = \lim_{N\to+\infty}N\frac{2\sqrt{2}\sigma_u\sigma_v}{\sqrt{\pi}}\sqrt{N}\exp\left(-\frac{N}{2}\right) = 0.$$

Pricing Schedule:

From (1) we have :

$$P(w) = \widetilde{p}(w, N) = \frac{2\sigma_v}{\sqrt{2\pi}} \tanh\left(\frac{w\sqrt{N}}{\sigma_u}\right),$$

thus we obtain the following result:

if S=1,

(9) $$\lim_{N\to+\infty} P(w, N) = \frac{2\sigma_v}{\sqrt{2\pi}},$$

if S=-1,

(11) $$\lim_{N\to+\infty} P(w, N) = -\frac{2\sigma_v}{\sqrt{2\pi}}.$$

Liquidity:

$\lambda(w, N) = \dfrac{\partial p}{\partial w}(w)$ is the price pressure and corresponds to the inverse of the market depth:

$$\lambda(w, N) = \frac{2\sigma_v}{\sqrt{2\pi}\sigma_u} \frac{\sqrt{N}}{\left[\cosh(\frac{w\sqrt{N}}{\sigma_u})\right]^2}.$$

If $S = -1$,

$$\lim_{N\to+\infty} \lambda(w, N) = \lim_{N\to+\infty} \frac{2\sigma_v}{\sqrt{2\pi}\sigma_u} \frac{\sqrt{N}}{\left[\cosh(-\frac{w\sqrt{N}}{\sigma_u})\right]^2} = \lim_{N\to+\infty} \frac{2\sigma_v}{\sqrt{2\pi}\sigma_u} \frac{\sqrt{N}}{\left[\exp(-\frac{w\sqrt{N}}{\sigma_u})\right]^2},$$

if $S = 1$,

$$\lim_{N\to+\infty} \lambda(w, N) = \lim_{N\to+\infty} \frac{2\sigma_v}{\sqrt{2\pi}\sigma_u} \frac{\sqrt{N}}{\left[\cosh(\frac{w\sqrt{N}}{\sigma_u})\right]^2} = \lim_{N\to+\infty} \frac{2\sigma_v}{\sqrt{2\pi}\sigma_u} \frac{\sqrt{N}}{\left[\exp(\frac{w\sqrt{N}}{\sigma_u})\right]^2}.$$

Thus if $S = -1$ and $S = 1$

(12) $$\lim_{N\to+\infty} \lambda(w, N) = 0.$$