Abstract—In this paper we propose a scheme for output tracking of nonminimum phase flat systems. The technique, which is applicable even in the minimum phase case, uses an inverse trajectory for feedforward which is stabilized by a state tracker built using the flat output. In contrast to approximate-linearization based control [Hauser, J., S. Sastry and G. Meyer (1992)]. Nonlinear control design for slightly nonminimum phase systems: application to VTOL aircraft. *Automatica, 28*, 665–679, which yields good performance in slightly nonminimum phase systems, our approach applies equally well to slightly and strongly nonminimum phase systems. The proposed control scheme and the approximate-linearization based controller are applied to a Vertical Take Off and Landing (VTOL) aircraft example and the performances are compared.

1. Introduction

An important control problem is output tracking: given the input–output system

\[ x = f(x, u), \]

\[ y = h(x), \]

the aim is to build a controller such that \( y(t) \) tracks any prescribed reference trajectory \( y_r(t) \). To avoid ambiguities in the sequel, we call (1) the input-state (IS) system (i.e. without a defined output), (2) the tracking output, and (1)–(2) the input-output (IO) system.

When the IO system is minimum phase, the tracking problem is easily solved (at least locally) by IO linearization (see e.g. the work by Isidori (1989)). But this simple method cannot be satisfactorily applied to a nonminimum phase system: tracking by IO linearization requires integrating the internal dynamics, which is unstable in this case. It is important to acknowledge that the obstruction to exact trajectory tracking for nonminimum phase systems is of open-loop nature: in general, a state trajectory \( (x(t), u(t)) \) such that \( h(x(t)) = y(t) \) is unbounded. Since the internal dynamics is not changed by the feedback, exact tracking is contingent upon finding a suitable state trajectory.

One approach, for slightly nonminimum phase systems, is to approximate the system by a minimum phase one, as in Hirschorn’s (1979) work for minimum phase systems, but is noncausal (rather than unstable) in the nonminimum phase case, which results in initial condition mismatch.

We apply this approach to the simplified VTOL aircraft example considered by Hauser et al. (1992). Note that the feedback design relies heavily on the flatness of the system, and it is indeed a very strong structural property, but many (idealized) physical systems seem to be flat — and VTOL aircraft is one such flat system. In fact, any planar rigid body with two input forces acting at a point is flat (see, for example, Shus et al. (1995)), and hence the following can also be applied to such systems.

2. A simple planar VTOL aircraft

As considered by Hauser et al. (1992), we study a very simplified PVTOL (Planar Vertical Take Off and Landing)
Let \((i, \, j, \, k)\) be a fixed inertial frame, and \((i_b, \, j_b, \, k_b)\), with \(j_b = j\) be a moving frame attached to the aircraft (body axes). The forces acting on the system are

\[
T = T_{k_b},
\]
\[
F_1 = (\sin \alpha_{i_b} + \cos \alpha_{i_b} k)F,
\]
\[
F_2 = (\sin \alpha_{i_b} - \cos \alpha_{i_b} k)F,
\]
\[
mg = -mg_k.
\]

The equation of motion is written in terms of the center of mass \(C\) as

\[
m_{\bar{C}} \dot{\alpha} = T + F_1 + F_2 + mg_k,
\]
\[
\sigma_C = CM_1 \times F_1 + CM_2 \times F_2,
\]

where \(\alpha_{i_b}\) is the acceleration of \(C\) and \(\sigma_C\) is the angular momentum about \(C\). \(M_1\) and \(M_2\) are the points at which the forces \(F_1\) and \(F_2\) are located. Expanding these equations gives

\[
m(\ddot{x} + \ddot{y}) = T_{k_b} + 2F \sin \alpha_{i_b} k_b - mgk,
\]
\[
-\dot{\theta} \dot{\phi} = -2F \cos \alpha_{i_b} j_b.
\]

where \(J\) is the moment of inertia about \(C\), and \(l\) the distance from \(C\) to points \(M_1\) and \(M_2\). Setting

\[
u_1 := \frac{T}{m},
\]
\[
u_2 := \frac{2F}{m} \cos \alpha,
\]
\[
\varepsilon := \tan \alpha,
\]
\[
\lambda := \frac{ml}{J},
\]

and projecting into the fixed frame, we finally get

\[
\ddot{x} = -\nu_1 \sin \theta + \nu_2 \cos \theta,
\]
\[
\ddot{z} = \nu_1 \cos \theta + \nu_2 \sin \theta - g.
\]

Like in the work of Hauser et al. (1992), we will normalize \(g\) and \(\lambda\) to 1 in the simulations.

3. Searching for a flat output

We investigate the structure of the system for a whole family of outputs representing points which are fixed with respect to the VTOL aircraft body (see Fig. 3). Their coordinates in the fixed frame are

\[
y := (x + a \cos \theta - b \sin \theta, z + a \sin \theta + b \cos \theta).
\]

where \(a, b\) are parameters. The restriction to this family is motivated by physical intuition: since the system is a rigid body, it is not unreasonable to think that a flat output, provided it exists at all, should have a physical meaning and thus should correspond to points moving with the aircraft. Moreover, this leads to much simpler computations. To compute the inverse dynamics of the system, we have to differentiate the output until both controls appear independently. The first time derivative of the output is given by

\[
\ddot{y}_1 = \dot{y}_1 = -a \dot{\theta} \sin \theta - b \dot{\theta} \cos \theta
\]
\[
\ddot{y}_2 = \dot{y}_2 = z + a \dot{\theta} \cos \theta - b \dot{\theta} \sin \theta.
\]

As no control appears, we continue to differentiate

\[
\dddot{y}_1 = -\zeta \sin \theta + \left(\varepsilon - \lambda \dot{\theta}\right) \nu_2 - a \dot{\theta}^2 \cos \theta,
\]
\[
\dddot{y}_2 = \zeta \cos \theta + \left(\varepsilon - \lambda \dot{\theta}\right) \nu_2 - a \dot{\theta}^2 \sin \theta - g.
\]

where \(\zeta := \nu_1 - b \dot{\theta}^2 + \lambda \nu_2\). When \(b = \varepsilon / \lambda\), both controls appear independently, hence the structure at infinity is \([2, 2]\) and the origin is a regular point (see the work by Martin (1993), and the references therein, for the definition of structure at infinity and regularity used here). Thus, the internal dynamics (with dimension \(6 - (2 + 2) = 2\)) is given by
Fig. 4. Phase portrait for $a = 0, b = 0$.

Fig. 6. Phase portrait for $a = 0, b = 1$.

Fig. 5. Phase portrait for $a = 0.1, b = 0$.

Fig. 7. Phase portrait for $a = 0.1, b = 1$.

\[
\dot{\theta} = \frac{\lambda}{\epsilon - \lambda b} \left[ g \sin \theta + a \dot{\theta}^2 + (\dot{y}_1 \cos \theta + \dot{y}_2 \sin \theta) \right].
\]  

(7)

Depending on the sign of $\epsilon - \lambda b$ and $\alpha$, the zero dynamics (i.e. the internal dynamics (7) with $y_1$ and $y_2$ identically zero) has different qualitative behavior. In particular the IO system is strictly nonminimum phase when $b < \epsilon/\lambda$ (since $\epsilon$ and $\lambda$ are positive) and weakly nonminimum phase (see, for example, Hauser et al. (1992)) when $b > \epsilon/\lambda$. We show various phase portraits ($\theta$ vs $\dot{\theta}$) for different values of $a, b$, with $\epsilon = 0.5$ and $\lambda = 1.0$. When $b < \epsilon/\lambda$ and $a = 0$, the phase portrait (Fig. 4) is symmetric, and the equilibrium point $(0, 0)$ is a saddle. The phase portrait is smoothly deformed by the $a\dot{\theta}^2$ term, and $(0, 0)$ is still a saddle when $a \neq 0$ (Fig. 5). When $b > \epsilon/\lambda$ and $a = 0$, $(0, 0)$ is a center (Fig. 6). When $a = 0$, the phase portrait is again smoothly deformed by the $a\dot{\theta}^2$ term, and $(0, 0)$ is still a center (Fig. 7).

We now resume the computations on the IO system. When $b = \epsilon/\lambda$, the controls appear in a dependent form (in $\dot{y}$) and so we differentiate once more:

\[
\begin{align*}
\dot{y}_1^{(3)} &= -\zeta \sin \theta - \zeta \dot{\theta} \cos \theta + a \dot{\theta}^2 \sin \theta - 2u_2 a \lambda \dot{\theta} \cos \theta, \\
\dot{y}_2^{(3)} &= \zeta \cos \theta - \zeta \dot{\theta} \sin \theta - a \dot{\theta}^2 \cos \theta - 2u_2 a \lambda \dot{\theta} \sin \theta.
\end{align*}
\]

Note that if $a = 0$, then the second control appears but with a $\theta$ factor. The generic structure at infinity is $[2, 3]$, but the tangent structure around the origin is $[2, 4]$. The system is not regular at the origin, and the internal dynamics is not well defined. So we consider the case when $a = 0$. Since the second control is absent we differentiate again:

\[
\begin{align*}
\dot{y}_1^{(4)} &= -\zeta \sin \theta - u_2 \zeta \cos \theta - 2\zeta \dot{\theta} \cos \theta + \zeta \dot{\theta}^2 \sin \theta, \\
\dot{y}_2^{(4)} &= \zeta \cos \theta - u_2 \zeta \sin \theta - 2\zeta \dot{\theta} \sin \theta - \zeta \dot{\theta}^2 \cos \theta.
\end{align*}
\]

The structure at infinity is now $[2, 4]$ and the origin is a regular point at the origin $\zeta = g$. Thus the IO system has no zero dynamics when the output is
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\[ X_1 = x - \frac{\xi}{\lambda} \sin \theta, \quad X_1 = x - \frac{\xi}{\lambda} \theta \cos \theta, \quad X_1 = -\xi \sin \theta, \]
\[ X_{1}^{(2)} = -\xi \sin \theta - \xi \theta \cos \theta \]
\[ X_2 = z + \frac{\xi}{\lambda} \cos \theta, \quad X_2 = \frac{\xi}{\lambda} \theta \sin \theta, \quad X_2 = -\xi \cos \theta \quad \text{and \ dynamic feedback} \]
\[ u_1 = z + \frac{\xi}{\lambda} \theta^2, \quad u_2 = \frac{\xi}{\lambda} \sin \theta, \]
\[ \xi = -v_1 \cos \theta + v_2 \cos \theta + \xi \theta^2 \]
transforming (3) into
\[ X_1^{(4)} = v_1, \quad X_2^{(4)} = v_2. \]

The feedback and the coordinate change are well defined when \( \xi = 0 \), i.e. \( \omega = \frac{\xi}{\lambda} \theta^2 \). Placing the poles of the linear system (9), \( \omega \) can be easily constructed an exponentially stable state tracker. Note that this is just an output tracker for the flat output \( \chi \).

The closed-loop problem is thus solved. The issue now is to solve the open-loop problem, i.e. to generate a reference state trajectory (or equivalently a reference for the flat output) from the reference of the tracking output (or rather the output which is to be tracked). Depending on the tracking output, the trajectory generator (Fig. 1) may be more or less complex.

As in the work of Hauser et al. (1992) on VTOL aircraft, the tracking output is the center of mass \( (a = b = 0) \), which results in a slightly \( (\epsilon \) is small) nonminimum phase system. Notice nevertheless that if the flat output were the tracking output, the problem would be solved (the trajectory generator is trivial in this case). Indeed, for the VTOL aircraft, one might ask why the center of oscillation is not a good choice of a tracking output as the center of mass (since \( \epsilon / \lambda \) is in general small, the two points are very close). Indeed, the choice of a tracking output is in general only a translation of a broader engineering problem of integrated control/system design with a freedom in the choice of outputs. Notice also that in a take-off or landing problem, a point located between the wheels (point \( W \) in Fig. 3) may be a more relevant choice of a tracking output, which leads to a strongly nonminimum phase system. These remarks are worth remembering, because in many other applications, the same control problem may allow a similar freedom in the choice of the tracking output.

We now return to the design of the trajectory generator when the tracking output is the center of mass. Since the center of mass \( C \) and center of oscillation \( H \) are very close, we can easily construct an approximate trajectory generator as follows. Let \( y_{\text{t}}(t) \) be a reference trajectory for the center of mass. Then
\[ x_{\text{t}}(t) := \left( y_{\text{t}}(t), y_{\text{t}}(t) + \frac{\xi}{\lambda} \right) \]
and its derivatives can be used as a reference to the center of oscillation and the tracking error is
\[ \| x_{\text{t}} - y_{\text{t}} \| \leq \frac{\xi}{\lambda} \| \sin \theta \|, \]
\[ \| x_{\text{t}} - y_{\text{t}} \| \leq \frac{\xi}{\lambda} (1 - \cos \theta). \]

This approximation is good provided \( \epsilon / \lambda \) is small. This is in fact an order zero correction, and higher-order corrections may be used along the same lines. Rather than going further in

4. The control scheme

As a consequence of its flatness, the IS system is fully linearizable by (dynamic) feedback. Indeed, we get from the previous computations a coordinate change
\[ (x, x, z, z, \theta, \zeta, \zeta) \rightarrow \left( X_1, X_1, X_2, X_2, X_2, X_2, X_2^{(3)} \right) \]
defined by

Fig. 8. Nature of zero dynamics.

\[ \chi = (x - \frac{\xi}{\lambda} \sin \theta, z + \frac{\xi}{\lambda} \cos \theta). \]

In other words the IS system is flat (see, for example, Fliess et al. (1992), Fliess et al. (1995) and Martin (1992)), i.e. its state and input can be expressed as functions (explicitly defined in the next section) of \( \chi \) and its derivatives, and \( \chi \) is a flat output. This is easily seen, since from equation (6) we get
\[ (X_1 - x)^2 + (X_2 - z)^2 = \left( \frac{\xi}{\lambda} \right)^2, \]
\[ \dot{X}_1 = 0. \]
\[ X_1 \cos \theta - (X_2 + y) \sin \theta = 0. \]

These relations give \( x, z \) and \( \theta \) as functions of \( \chi \) and its derivatives (up to order 2). Similarly, by differentiating \( x, z, \theta, \) and \( u \) can also be obtained as functions of \( \chi \) and its derivatives (up to order 4). There is thus a one-to-one correspondence between solutions \((x_0(t), u_0(t))\) of system (3) and any given \( x_0(t) \).

Remark 1. The above relations correspond to a pendulum of length \( \epsilon / \lambda \) (Martin (1992)), and as noticed by Fliess et al. (1992), the flat output has been known (but for other reasons) to mechanicians for a long time as the Huygens center of oscillation.

Remark 2. No general method to check whether a system is flat is known, but it is often possible to find, as in this case, a flat output by physical reasoning.

To conclude, we depict the nature of zero-dynamics for different choices of outputs in Fig. 8. If we consider as output a point, fixed to the aircraft body and located above the solid line, then the equilibrium point, \((\theta_0, \delta_0) = (0, 0)\) of the zero dynamics is a center. For points selected as outputs that are below the solid line, the equilibrium point becomes a saddle (nonminimum phase). For points chosen as outputs along the solid line the system is degenerate, except at \( H \) (which is a flat output) where we have a trivial case of no zero dynamics.
this direction, we will concentrate on a more general method for trajectory generation, which does not rely on the fact that the flat output is close to the tracking output, and which applies equally well to strongly and slightly nonminimum phase systems.

5. Generating bounded state trajectories

We investigate here the generation of a suitable reference state trajectory when the tracking output is a fixed point of the aircraft (4). We exclude degenerate points (i.e. \( b = \varepsilon/\lambda \) and \( a \neq 0 \)), so that the internal dynamics is well defined. For simplicity we assume the reference output trajectory \( y_{r}(t) \) is given on some interval \([0, T]\) with its acceleration \( \dot{y}_{r}(t) \) zero outside this interval. In other words, we consider motions of the aircraft where the output has a constant velocity at the beginning and end. If we have a solution \((\theta_{i}(t), \dot{\theta}_{i}(t))\) to the internal dynamics (7), we can easily find a state trajectory \( x_{s}(t), u_{s}(t) \) which exactly tracks the output (i.e. \( h(x_{s}(t)) = y_{r}(t) \)) as a function of \( y_{r}(t), \dot{y}_{r}(t), \dot{\theta}_{i}(t), \ddot{\theta}_{i}(t) \) by inverting equations (4)-(6).

Thus, if we integrate the internal dynamics starting from some initial conditions \((\theta_{i}(0), \dot{\theta}_{i}(0))\) we get a reference state trajectory, which depends of course on these initial conditions. In the minimum phase case, such a trajectory is always bounded, but this is no longer true in the nonminimum phase case (from here on, we mean bounded in the \( \| \cdot \|_{1,\omega} := \| \cdot \|_{1} + \| \cdot \|_{\infty} \) norm; hence a trajectory for which \( \theta \) spins forever is unbounded in this sense). If tracking is only required for a "short" interval of time, then in practice an unbounded state trajectory may be acceptable as a reference. However, for large intervals of time, it leads to an unacceptable behavior. Notice that it is unwise, for this reason, to choose a point above the center of oscillation as the tracking output: the system in this case is only weakly minimum phase and nearly any reference \( y_{r}(t) \) produces unbounded state trajectories. But, then the tracking output is below the center of oscillation, the system is strictly nonminimum phase and we have the following result.

**Theorem 1.** If the internal dynamics has a hyperbolic equilibrium at \((0, 0)\), then there exists a bounded solution to the internal dynamics (7), provided \( \| \cdot \|_{1,\omega} \) is not too large. This solution is locally unique and satisfies \( \theta_{i}(\pm \infty) = \dot{\theta}_{i}(\pm \infty) = 0 \).

From this solution, we generate as above the desired state trajectory \( x_{s}, \) which satisfies \( h(x_{s}(t)) = y_{r}(t) \). Note that in general the reference \((\theta_{i}(0), \dot{\theta}_{i}(0))\) will be different from the actual \((\theta_{0}, \dot{\theta}_{0})\). Hence, in contrast with the minimum phase case, there will be in general an initial condition mismatch.

We refer the reader to the work by Devasia et al. (1993) for a detailed proof in the general case. The idea of finding this bounded solution is to solve a sequence of linear ordinary differential equations

\[
\eta_{n+1}(t) = A\eta_{n}(t) + \left[ s(\eta, \psi) - A\eta(t) \right]
\]

with boundary conditions \( \eta(\pm \infty) = 0 \), starting from \( \eta_{0} := 0 \); here \( \eta := \left( \theta, \dot{\theta} \right) \), the internal dynamics (7) has been rewritten as \( \eta = s(\eta, \psi), \) and \( A \) is the linearization of

\[
A := \begin{bmatrix}
0 & 1 \\
\psi^{2} & 0
\end{bmatrix}
\]

with \( \psi^{2} := \frac{\lambda}{2\pi} \). Note that the term in brackets in (10)

\[
s(\eta, \psi) - A\eta(t) := \begin{bmatrix} 0 \\ ps_{n}(t) \end{bmatrix}
\]

can be viewed as an additive perturbation on the linearized internal dynamics. Like the internal dynamics itself, the linear equation (10) also has an unstable equilibrium point. We explicitly find a bounded solution by first decomposing (10) into stable and unstable subsystems

\[
\dot{z}_{e} = -y_{e} - \frac{ps_{n}(t)}{2y}, \quad \dot{z}_{e} = y_{e} + \frac{ps_{n}(t)}{2y}
\]

where

\[
\begin{bmatrix}
z_{e} \\
z_{e}
\end{bmatrix} = \begin{bmatrix}
y_{e} \quad 1 \\
-1 \quad 2y
\end{bmatrix} \eta,
\]

and then flowing along the stable manifold forward in time, and backward along the unstable manifold to obtain the only bounded solution as

\[
z_{e}(t) = -\frac{1}{2y} \int_{-\infty}^{t} e^{-\tau(y_{e}')} ps_{n}(\tau) d\tau,
\]

\[
z_{e}(t) = -\frac{1}{2y} \int_{-\infty}^{t} e^{\tau(y_{e}')} ps_{n}(\tau) d\tau.
\]

It can be shown that \( \eta_{n} \) converges, as \( n \to \infty \), to a bounded solution of the internal dynamics. In fact the algorithm corresponds to repeatedly applying a contraction, whose fixed point is a solution of the internal dynamics.

6. Simulations

We apply our method to a model PVTOL aircraft and compare the results with the approximate linearization design in the work by Hauser et al. (1992). As shown by Hauser and co-workers the approximate linearization design yields good results when the coupling factor \( \varepsilon \) is small, however the performance degrades as \( \varepsilon \) increases. To show that we are not restricted to slightly nonminimum phase systems, we choose a relatively high \( \varepsilon = 0.5 \) and track the center of mass (\( a = b = 0 \)). This is equivalent to tracking a point far below the center of oscillation. The maneuvers considered are from level to level flight (i.e. constant velocity of the center of mass and \( \theta = \dot{\theta} = 0 \)) at the beginning and end of the maneuvers. The simulation results are shown in Fig. 9.

To illustrate the robustness of the scheme, we designed the state tracker and generator with \( \lambda = 1 \) as a nominal value, and performed the simulations with different \( \lambda \) values in the aircraft model. Remember that \( \lambda \) is an inherent parameter, which is not necessarily well known in practice. With our control scheme, a \( \pm 15\% \) error in \( \lambda \) only slightly affects the tracking performance. In contrast (see Fig. 10) the approximate-linearization-based design tracks lateral motions very poorly, and is affected much more by disturbances in \( \lambda \). We note again that this low performance (with the approximation-based control scheme) is present due to the large values of \( \varepsilon \), and the performance of the two schemes are similar for low values of \( \varepsilon \).

We have used gains that achieve tracking comparable (see Fig. 11) with the performance of the controller proposed by Hauser et al. (1992).

7. Conclusion

We developed a control scheme for output tracking of nonminimum phase flat systems. The tracking problem is split into designing (1) a state tracker using the flat output (solved in Section 4) and (2) a trajectory generator to feed the state tracker (solved in Section 5). This scheme was applied to a Vertical Take Off and Landing (VTOL) aircraft example yielding a substantial performance improvement over the approximate-linearization-based control scheme (see simulations in Section 6). The approximate-linearization-based controller approximates the nonminimum phase system by a minimum phase system, and hence its performance degrades if the system is strongly nonminimum phase (for example, the...
Fig. 9. Tracking with inversion-based controller $\lambda$ varying.

Fig. 10. Tracking with approximate-linearization-based controller $\lambda$ varying.
VTOL aircraft model with a large nonlinear coupling factor). In contrast, our scheme does not require such approximations and is applicable equally well to strongly and slightly nonminimum phase systems.

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