Probabilistic views on growth-fragmentation equations [1]

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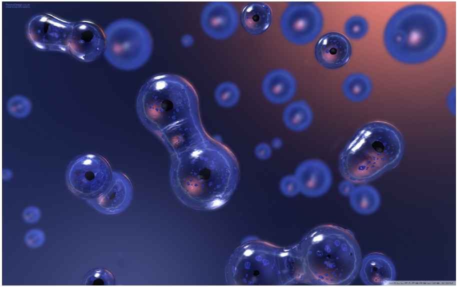
[1] Partly based on joint works with Alexander Watson

Growth-fragmentation equations

The growth-fragmentation equation is a linear evolution equation that describes a system of growing and dividing particles (cells, bacteria, ...)



Represent the concentration of particles with size $y > 0$ at time $t \geq 0$ by a measure

$$\mu_t^x(\mathrm{d}y),$$

assuming for simplicity that one starts at time $t = 0$ from a unit concentration of particles of size $x > 0$,

$$\mu_0^x = \delta_x.$$

The kernel $\mu_t^x(\mathrm{d}y)$ yields the operator

$$T_t f(x) = \langle \mu_t^x, f \rangle = \int_0^\infty f(y) \mu_t^x(\mathrm{d}y),$$

and the growth-fragmentation equation has the form

$$\frac{\mathrm{d}T_t f}{\mathrm{d}t} = \mathcal{A} T_t f.$$

In words, the operators $(T_t)_{t\geq 0}$ form a semigroup with infinitesimal generator

$$\mathcal{A}f(x) = c(x)f'(x) + \int_0^x f(y)k(x,y)\mathrm{d}y - K(x)f(x).$$

- c is the growth rate of particles,
- $k(x,y)$ is the rate of creation of $y > 0$ by dislocation of $x > y$,
- Conservation of mass during dislocation

$$xK(x) = \int_0^x yk(x,y)\mathrm{d}y.$$

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Usually the semigroup cannot be found explicit in terms of the infinitesimal generator.

Nonetheless information about its large time behavior has be obtained using spectral techniques for operators by many authors in the PDE community and over many years.

(See e.g. Perthame & Ryzhik, Doumic and Gabriel, Cáceres, Cañizo & Mischler, Doumic & Escobedo, ...)

Background on asymptotic behavior of semigroups

Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of positive linear operators (not necessarily contractions) :

$$T_s T_t = T_{s+t}$$

with infinitesimal generator $\mathcal{A}$:

$$\lim_{t\to 0+} \frac{T_t f - f}{t} = \mathcal{A} f \qquad \text{for } f \in \mathcal{D}_{\mathcal{A}}.$$

Informally,

$$T_t = \mathrm{e}^{t\mathcal{A}}.$$

Perron-Frobenius / Krein-Rutman paradigm:

Under *appropriate* assumptions, there exists a principal eigenvalue $\lambda \in \mathbb{R}$ for both $\mathcal{A}$ and the dual operator $\mathcal{A}'$.

The eigenfunctions h and m

$$\mathcal{A}h = \lambda h \qquad \mathcal{A}'m = \lambda m$$

can be chosen positive and normalized such that $\langle m, h \rangle = 1$.

Then

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t} T_t f(x) = h(x)\langle m, f \rangle.$$

Convergence is exponentially fast when there is a spectral gap.

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For contraction semigroups ($\|T_t f\| \leq \|f\|$), one can associate a Markov process $(X_t)_{t\geq 0}$ such that

$$T_t f(x) = \mathbb{E}_x(f(X_t)).$$

When X is irreducible positive recurrent, there is a unique stationary law m, i.e.

$$mT_t = m \text{ and } \langle m, \mathbf{1}\rangle = 1.$$

If X is further aperiodic, convergence to equilibrium holds:

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This is merely a probabilistic reformulation of the asymptotic behavior of the semigroup with

$$\lambda = 0\ ,\ \ h(x) \equiv 1\ ,\ \ \mathcal{A}h = 0\ ,\ \ \mathcal{A}'m = 0.$$

The probabilistic approach provides criteria (Foster-Lyapunov) for ergodicity or exponential ergodicity of X, and stochastic representations of the stationary law m (ergodic average, or via the occupation measure of the excursion).

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Feynman-Kac representation

The growth-fragmentation semigroup $(T_t)_{t\geq 0}$ is not contractive.

Nonetheless, under the assumption that the growth rate is sublinear:

$$\sup_{x>0} c(x)/x < \infty,$$

it can be related to a Markov process

$$X = (X_t)_{t\geq 0}.$$

Specifically,

$$\mathcal{G}f(x) = c(x)f'(x) + \int_0^x (f(y) - f(x))\frac{y}{x}k(x,y)\mathrm{d}y,$$

is the infinitesimal generator of a Markov process that describes the evolution of a 'distinguished' particle:

At each dislocation event, the child particle is selected in its sibling by random size-biased sampling.

Writing $\bar{f}(x) = xf(x)$, we have

$$x^{-1}\mathcal{A}\bar{f}(x) = \mathcal{G}f(x) + \frac{c(x)}{x}f(x).$$

The growth-fragmentation semigroup has the Feynman-Kac representation:

$$T_t f(x) = x\mathbb{E}_x\left(\frac{f(X_t)}{X_t}\mathcal{E}_t\right)$$

with

$$\mathcal{E}_t = \exp\left(\int_0^t \frac{c(X_s)}{X_s}\mathrm{d}s\right).$$

The Feynman-Kac solution is given in terms of the location X_t of the Markov process at time t, and the weight $\mathcal{E}_t$ that depends on the whole trajectory of $(X_s : 0 \leq s \leq t)$.

In the framework of branching processes, this would be called a many-to-one formula: it expresses the expectation of a linear functional of the particle system in terms the trajectory of a distinguished particle.

Overview of results

The asymptotic behavior of (T_t) can be studied via regeneration for passage times at a given state.

Introduce the first hitting time of $x > 0$ by X

$$H(x) := \inf \{t > 0 : X_t = x\} .$$

When $X_0 = x$, $H(x)$ is the duration of the first excursion away from x.

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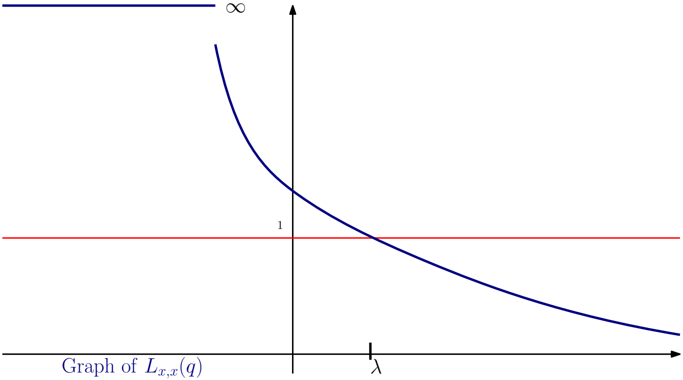
The Laplace transform

$$L_{x,y}(q) := \mathbb{E}_x\left(\mathrm{e}^{-qH(y)}\mathcal{E}_{H(y)}, H(y) < \infty\right), \qquad q \in \mathbb{R}.$$

plays a fundamental role.

For every x, y fixed, $L_{x,y} : \mathbb{R} \to (0, \infty]$ is a convex non-increasing and right-continuous function with

$$\lim_{q\to+\infty} L_{x,y}(q) = 0 \quad \text{and} \quad \lim_{q\to-\infty} L_{x,y}(q) = +\infty.$$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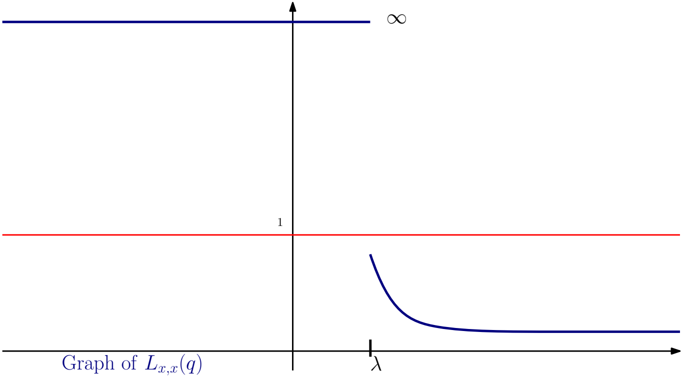
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Graph of $L_{x,x}(q)$

The Malthus exponent is then defined as:

$$\lambda := \inf\{q \in \mathbb{R} : L_{x,x}(q) < 1\}$$

(does not depend on $x > 0$).

We next fix $x_0 > 0$ arbitrarily, and set

$$\ell(x) = L_{x,x_0}(\lambda)\ , \ \bar{\ell}(x) := x\ell(x).$$

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Theorem (Malthusian behavior)

Assume

$$L_{x_0,x_0}(\lambda) = 1 \text{ and } L'_{x_0,x_0}(\lambda) > -\infty. \qquad (1)$$

Then for every $f \in \mathcal{C}_K^+$,

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t} T_t f(x) = \bar{\ell}(x)\langle m, f\rangle,$$

with

$$m(\mathrm{d}y) := \frac{\mathrm{d}y}{c(y)\bar{\ell}(y)|L'_{y,y}(\lambda)|}, \qquad y > 0.$$

Conversely, (1) *is always fulfilled when the Malthusian behavior holds even in some weak sense.*

By stochastic calculus, one can check that $\bar{\ell}$ is an eigenfunction with eigenvalue λ of (some extension of) $\mathcal{A}$:

$$\mathcal{A}\bar{\ell} = \lambda\bar{\ell}.$$

Under some mild technical assumptions, one can also check that m is an eigenmeasure with eigenvalue λ for the dual operator $\mathcal{A}'$:

$$\langle \mathcal{A}'m, f \rangle = \langle m, \mathcal{A}f \rangle = \lambda \langle m, f \rangle , \qquad f \in \mathcal{D}_{\mathcal{A}}.$$

Theorem (exponential ergodicity)

Assume

$$L_{x_0,x_0}(q) \in (1,\infty) \qquad \textit{for some } q \in \mathbb{R}. \tag{2}$$

Then (1) *holds and there exists* $\eta > 0$ *such that*

$$\mathrm{e}^{-\lambda t} T_t f(x) = \bar{\ell}(x)\langle m, f\rangle + o(\mathrm{e}^{-\eta t}).$$

In turn, (2) *holds whenever*

$$\limsup_{x\to 0+} c(x)/x < \lambda \textit{ and } \limsup_{x\to\infty} c(x)/x < \lambda. \tag{3}$$

Thanks to (3), fully explicit criteria can now be given in terms of the coefficients.

Notably, exponential ergodicity holds provided that X is recurrent and

$$\lim_{x\to 0+} c(x)/x = \lim_{x\to\infty} c(x)/x = \inf_{x>0} c(x)/x,$$

(because then $\lambda > \inf_{x>0} c(x)/x$).

We now turn our attention to growth-fragmentation processes, which is the stochastic version of growth-fragmentation semigroups.

Specifically, we consider a branching process $\mathbf{Z} = (\mathbf{Z}_t)_{t \geq 0}$ where individuals have a mass that grows continuously with rate c and break randomly inducing the fragmentation kernel $k(x, y)$.

The relation between the deterministic and the stochastic model is given

$$T_t f(x) = \mathbb{E}_x \left[\langle \mathbf{Z}_t, f \rangle \right],$$

where $\mathbb{E}_x$ refers to the mathematical expectation when the branching process $\mathbf{Z}$ starts at time 0 from a single ancestor with mass $x > 0$.

In the line of several classical results in branching theory (Nerman, Jagers, Biggins, ...), one obtains the following strong version of the previous theorems.

Theorem

Assume

$$\limsup_{x\to 0+} c(x)/x < \lambda \textit{ and } \limsup_{x\to\infty} c(x)/x < \lambda.$$

The process

$$W_t = \mathrm{e}^{-\lambda t}\langle \mathbf{Z}_t, \bar{\ell}\rangle, \qquad t \geq 0$$

is a martingale bounded in $L^2(\mathbb{P}_x)$ *for every* $x > 0$, *and for every continuous function* f *with compact support, we have*

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t}\langle \mathbf{Z}_t, f\rangle = \langle m, f\rangle W_\infty, \qquad \textit{in } L^1(\mathbb{P}_x).$$

Elements of proofs

The cornerstone of the proof consists in observing that

$$\mathcal{M}_t := \ell(X_t)\mathrm{e}^{-\lambda t}\mathcal{E}_t$$

is a martingale, and then using it for changing the probability measure into the law of a new process Y:

$$\mathbb{E}_x\left(\Phi(Y_s : 0 \leq s \leq t)\right) = \frac{1}{\ell(x)}\mathbb{E}_x\left(\Phi(X_s : 0 \leq s \leq t)\mathcal{M}_t\right).$$

The Feynman-Kac formula for the growth-fragmentation semigroup then takes the simpler form

$$T_t f(x) = x\mathbb{E}_x\left(\frac{f(X_t)}{X_t}\mathcal{E}_t\right) = \mathrm{e}^{\lambda t}\bar{\ell}(x)\mathbb{E}_x\left(\frac{f(Y_t)}{\bar{\ell}(Y_t)}\right).$$

So the study of its asymptotic behavior boils down to establishing convergence to equilibrium for the process Y.

The process Y is again Markovian with infinitesimal generator

$$\mathcal{G}_Y g(x) = \frac{1}{\ell(x)} \mathcal{G}(g\ell)(x) + (c(x)/x - \lambda) g(x).$$

We thus simply need to prove that Y is (exponentially) ergodic and then compute its stationary law.

Let

$$H_Y(x) = \inf\{t > 0 : Y_t = x\}$$

be the first hitting time of x by Y. Then

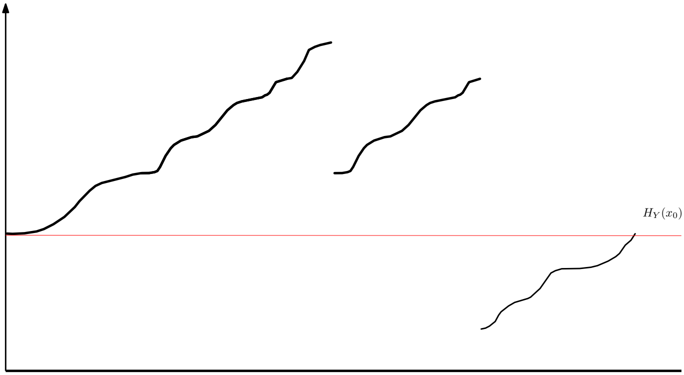
$$\mathbb{E}_x(\mathrm{e}^{-qH_Y(x)}) = L_{x,x}(\lambda + q).$$

In particular:

$$L'_{x,x}(\lambda) > -\infty \text{ iff } \mathbb{E}_x(H_Y(x)) < \infty \text{ (} Y \text{ positive recurrent),}$$

$$L_{x,x}(\lambda - \varepsilon) < \infty \text{ for some } \epsilon > 0 \text{ iff } Y \text{ is exponentially recurrent.}$$

Then the (normalized) occupation measure of the excursion of Y away from x_0, say ν, is known to be a stationary law for Y.



Excursion of Y

One finds

$$\nu(\mathrm{d}y) = \frac{\mathrm{d}y}{c(y)|L'_{y,y}(\lambda)|}, \qquad y > 0$$

and conclude the proof by convergence to equilibrium, i.e.

$$\lim_{t\to\infty} \mathbb{E}_x(f(Y_t)) = \langle \nu, f \rangle.$$

Summary of the results

We have obtained expressions in terms of a certain Markov process X, whose infinitesimal generator $\mathcal{G}$ bears a simple relation to the growth-fragmentation operator $\mathcal{A}$, for:

- the growth-fragmentation semigroup T_t,
- the Malthus exponent λ (i.e. principal eigenvalue),
- the corresponding eigenfunction h and dual eigenmeasure ν,
- conditions for the existence of a spectral gap,
- strong versions for the stochastic model.

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